MATRIX ANALYSIS OF ELASTIC STRUCTURES

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Submitted in partial fulfillment of
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PREFACE

Procedures are developed for the static analysis of highly complex structures which are composed of elastic elements rigidly joined together and subjected to forces applied at their extremities or junction points. Matrix notation is used throughout the development. It is assumed that there is no interference between members and that the constraints do not alter in type or number as the structure deforms. Linear elasticity is assumed and the results are valid only for "small" deformations. As in similiar studies, matrix equations are obtained which relate deflections and rotations at various pertinent points of the structure to generalized loads applied to the structure. This study is distinguished by the fact that the basic procedure involves building up the flexibility matrix for a complicated structure by combining the flexibility matrices for simpler substructures. Another distinguishing feature is that most of the resulting matrices are exhibited as sums of simpler matrices, the form and structure of which are readily evident.

A second and different point of view is also developed to yield a matrix, the order of which depends essentially on the number of junction points rather than the number of constrained anchor points as is usually the case.

This thesis was written at the U. S. Naval Postgraduate School, Monterey, California during the period October 1958

to May 1959. I am indebted to Professor John E. Brock for his continued patience, encouragement, and most capable guidance while acting as faculty advisor.

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CHAPTER 1

Introduction and Statement of the Problem

1.1. Introduction.

This thesis develops a method for describing and dealing with a complex elastic structure composed of elements which are rigidly joined together. The whole structure is fixed and supported by not less than the number of restraints required to preserve static equilibrium under the loads applied. It is assumed that the loads applied are of such magnitude that only "small" deformations result.

Having a structure, defined in terms of the location of joints and other points, also having the elastic, flexure properties of members or sections of the structure, the method developed in this thesis produces a system of linear equations. The equations give the displacement of pertinent points of the structure as functions of forces, or general loads, applied to the structure.

1.2. General Concepts.

Let us consider some of the properties of the elastic structure with which we will be dealing. It is composed of individual members and has certain points in it which will be of particular interest.

1. The individual members of the structure are such that under the forces applied there are only "small" elastic deformations. Each member is connected to other parts of the structure at not more than two points. These connection points are rigid joints. Usually the members will be beams,

filaments, or other structural members having one dominant dimension, but this is not required.

2. There are certain pertinent points in the structure where either

a) forces are applied to the structure, or

b) displacements are observed, or

c) there is a junction point where two or more members are rigidly joined.

Let us denote any one of these as a "pertinent point" of the structure. A particular pertinent point of the structure may serve as any or all of the above at the same time.

1.3. Specific Problem.

1.3.1. Assumptions.

1. We have the structure geometrically defined. That is, we know the position in space of each pertinent point of the structure when the structure is subjected to no external forces. This is the position from which we will measure small displacements.

2. We assume that we know the elastic, flexural properties of the individual members or sections of the structure. That is, we know what displacements each isolated member individually will have under a given loading, or in other words, we know the C matrix defined in section 2.1.3. This is a large assumption but it in no way affects the results obtained in this thesis. It is just a large problem in itself and this problem has been dealt with by others.[(1), (3) *]

*References are listed in the bibliography.

3. We assume that the number and nature of the restraints is not altered during the deformation.

1.3.2. Scope of Investigation.

By the analysis of several types of relatively simple structures we will attempt to find some general solutions to the problem at hand. We are essentially attempting to find an expression for the flexibility of the whole structure, stated in terms of the flexibility of individual members or sections and the location of pertinent points of members. As a result of the analysis in each particular case we will mathematically reduce the description of the structure under the given forces to a set of linear equations of the form

$$D_j = \sum_{i=1}^{n} A_{ji} F_i \qquad (i = 1,2,\ldots\ldots,n) \tag{1.3.2-1}$$

in which $j = 1, 2, \ldots, n$.

The method of this thesis is then to determine the coefficients A_{ji} in the various cases to be considered.

As a result of the analysis we have a set of linear equations relating the displacement of the various pertinent points in the structure to the forces imposed on the structure at various pertinent points. At this point we leave the problem for the algebraist to solve. There are many methods available to solve such a set of linear equations. In chapter 7 some attention is given to appropriate formulations and initial transformations which the algebraist may find it profitable to employ.

In chapter 8 an alternate point of view is explored for

describing a multianchor structure. In the method described in chapter 8, the number of equations in the resulting set depends only on the number of junction or branch points in the structure and does not depend on the number of members which are "fixed".

1.4. Background.

There is in the literature a wealth of material describing the analysis of a particular type of structure or complication in structure. Gabriel Kron has written many papers on the subject of the solution of complex networks. Elastic structures may be considered as such a network. In one of his papers[4] is presented a systemmatic procedure for solving very large elastic structures. When the results of this thesis are stated in general matrices, the form of the matrix equation is similiar to results stated by Kron. In this sense one might consider that this thesis presents in detail some of the results alluded to by Kron.

Both Brock[1] and Peck, et. al.[2] cover the concept of the transfer of forces and displacements in a structural system. The concept is introduced and some of its consequences explored in sections 2.1.5 and 2.1.6.

In the assumption of the problem we assumed that the elastic, flexural properties of a beam or filament were known. To determine these flexural properties of a beam or filament is a problem in itself. There are many papers available on the subject. Brock's papers[1], [3] develop methods to determine the flexibility coefficients of certain types of members.

It is tacitly assumed that the solution of any complex structure will involve the use of a high speed computer since the number of equations to be solved simultaneously will be quite large. The manipulation of the various flexibility matrices used in setting up the set of linear equations is most easily handled by computer techniques. The inversion of matrices, and matrix multiplication and addition will be involved. However, fundamentally each matrix inversion or multiplication involves only 6×6 or 6×1 matrices.

1.5. Notation.

In dealing with a complicated system it is realized that some consistent type of notation must be used in order for the reader to pass from section to section without undue difficulty. An attempt is made to use a notation which is general enough to be usable in the more complicated cases and yet can be simplified by the ommission of some subscripts in simpler problems.

One source of confusion is eliminated by having only one frame of reference, that is, all points are in the same coordinate system. The notation used in this thesis differs slightly from that used by Brock(1) in which his development allows each member to have its own associated coordinate system. However, if all these coordinate systems become one system the differences become immaterial.

In utilizing the results of this thesis the user would undoubtedly calculate the individual flexibility matrices in

a coordinate system of convenience and then make a general transformation of coordinates. Since the utility of the results obtained in the thesis depend on digital computers such as the IBM 704 or CDC 1604, such general transformations of coordinate systems are only a flicker of the computer's console lights.

1.5.1. Notation of Equations.

Since this thesis contains many equations which are referred back to from time to time, the equations are sequentially numbered within sections and are denoted both by section and sequential number. In some cases a suffix number is added when an equation is merely rewritten in an equivalent form or is intimately related to the basic equation. Equations taken from the appendix are prefixed with the letter I.

1.6. The Polyped.

In order to consider the structure we may have to disjoin it into smaller substructures. This process could be extended to disjoining the structure into individual members; however, we shall not normally disjoin the structure that far.

Let us disjoin the large structure into subsystems such that in each subsystem one end of each member is at a common point. That is the subsystem looks like a spider -- many legged. Thus we coin the term "polyped", which is formally defined in section 2.3.

It is possible to disjoin any complex structure into a

number of such polypeds. These polypeds may be simply linked to one another in a chain of polypeds or several members may link from one polyped to another in the chain of polypeds. Another possible situation is to have the chain of polypeds loop back on itself. Each of these types of polyped configuration is analyzed.

The overall structure is considered to be in equilibrium; however, it may be manyfold redundant. In considering each substructure of the structure we will consider it fixed at some point in order to discuss it and apply forces to it without having the substructure depart.

CHAPTER 2

Basic Definitions and Discussion of Simple Polypeds

2. In this chapter we will state the basic definitions and then derive further equalities and relationships which will be needed later on. The relationships are applied first to simple members and simple polypeds. Then various complications are introduced, i.e., multiple connected ends, multiple anchors, etc. Finally the important equations are generalized and a general notation is introduced.

In this chapter and throughout the rest of the thesis we will be dealing with matrices. In appendix I is a brief summary of matrix algebra including operations and properties of matrices which will be needed in this thesis. (Equations of appendix I are referred to by a designating number prefixed by the letter I). Further information on matrices may be found in a standard work on matrices.[6]

2.1. Definitions.

2.1.1. Forces.

Let $F = \{f_1, f_2, f_3, f_4, f_5, f_6\}$

be a column matrix describing the forces and moments applied to a structure at a point. The components f_1, f_2, and f_3 are the positive forces acting in the x, y, and z, directions respectively; the components f_4, f_5, and f_6 are the moments acting (in the positive sense of the right hand rule) about the x, y, and z axes respectively.

Let

$F(J,U) =$ the general 6 dimensional force acting <u>on</u> member J <u>at</u> point U. (2.1.1-1D)

2.1.2. Displacements.

Let $D = \{d_1, d_2, d_3, d_4, d_5, d_6\}$

be a column matrix describing the deflections which may be experienced by an infinitesimal element of structure. The components d_1, d_2, and d_3 are linear displacements in the x, y, and z directions respectively; the components d_4, d_5 and d_6 are rotations (positive according to the right hand rule) about the x, y, and z axes respectively. It is implicit here, and later, that the present analysis is valid only for small deflections.

A member or a point in a structure has a certain position in space when the member or structure has no external forces acting on it. This is the reference position of that point. Now let

$D(U) =$ the general 6 dimensional displacement of the point U from the reference position.

(2.1.2-1D)

2.1.3. Flexibility Matrix C.

Consider a particular member of the structure, say J, having ends at U and V, and for the moment consider that the end V is rigidly fixed in position. We are interested in the 6 x 6 square matrix C(J,U) such that

$$D(U) = C(J,U)\, F(J,U) \qquad (2.1.3\text{-}1D)$$

That is, a matrix relating the forces and moments applied at U and the displacements observed at U when the other end of the member, V, is held fixed. The C matrix is a matrix of influence coefficients.

$c_{ji}(J,U)$ the j-th component of deflection at U due to the application of a unit i-th component of force at U on member J.

(2.1.3-2D)

Thus for example, $c_{53}(J,U)$ is the positive rotation about the y axis of point U due to the application of a unit positive z direction force on member J at U, while the opposite end of the member, point V, is held fixed.

2.1.4. Properties of the Flexure Matrix C .

Because of the reciprocal theorems of structural theory, we have

$$c_{ij} = c_{ji} \qquad (2.1.4\text{-}1)$$

which may be written as

$$C(J,U) = C(J,U)^T \qquad (2.1.4\text{-}2)$$

Since we are dealing with a structure composed of passive elements, the flexure matrix is positive definite. This is also true of the larger flexure matrices which will be developed later in the thesis. This fact is not germane to the development but is useful in checking results and may serve to guide the selection of appropriate inversion processes

for computer application.

2.1.4.1 Ideal Members.

(This topic may be omitted on first reading). The term ideal member means that under the action of a force in a certain one of the six principal directions, no displacements result in any of the six directions. The flexure matrix of an ideal member has all zeros in the row, and column, corresponding to the direction of the idealization, since in idealizing we neglect axial deformation and the deformation due to cross shear, etc. For example, a member such that a force in the x-direction and a moment about the y-axis each produce no displacements or rotations at all is considered to be idealization in two dimensions. The flexure matrix for such a member is

$$C = \begin{bmatrix} e_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{22} & c_{23} & c_{24} & 0 & c_{26} \\ 0 & c_{32} & c_{33} & c_{34} & 0 & c_{36} \\ 0 & c_{42} & c_{43} & c_{44} & 0 & c_{46} \\ 0 & 0 & 0 & 0 & e_5 & 0 \\ 0 & c_{62} & c_{63} & c_{64} & 0 & c_{66} \end{bmatrix} \qquad (2.1.4\text{-}3)$$

In order to manipulate this matrix we will regard the diagonal terms e_1 and e_5 as infinitesimals. The C matrix is thus positive definite. (Even if the infinitesimals were zero the matrix is non-negative definite).

For the rest of this discussion we will consider the rows and columns of zeros to be concentrated at the top and

left of the C matrix. If this is not the case, they can be mathematically shifted there if we premultiply by a matrix N, where N is a unit matrix with its i-th row transferred to the first row, assuming that the i-th row is the one to be shifted. After inverting we must postmultiply the results by the matrix N to shift the rows back to the original positions.

Let us write the C matrix in a partitioned form

$$C = \begin{bmatrix} e & 0 \\ 0 & z \end{bmatrix} \qquad (2.1.4\text{-}4)$$

in which e is a matrix of infinitesimals

$e = \epsilon I$ (ϵ is a scalar infinitesimal)

I is of order j

and z is a nonsingular matrix of order 6 - j.

If we use equation I3.9-5 the inverse of this partitioned matrix is

$$C^{-1} = \begin{bmatrix} e^{-1} & 0 \\ 0 & z^{-1} \end{bmatrix} \qquad (2.1.4\text{-}5)$$

z^{-1} exists and is well defined since z is nonsingular

e^{-1} is a diagonal matrix of very large numbers.

Thus the inverse of the flexure matrix of an ideal member is not usable in this form. Fortunately when the inverse occurs in one of our final results, it occurs in the form of the inverse of the sum of the inverses. Let us consider an example.

$$C(EQ) = \left[C(I)^{-1} + C(A)^{-1} + C(B)^{-1} \right]^{-1} \qquad (2.1.4\text{-}6)$$

In this example I is an ideal member and A and B are ordinary physical members. We will partition the flexure matrices of the ordinary matrices in the same manner as the flexure matrix for the ideal member. Let us denote the submatrices of the inverse matrices with an overscore. For the ordinary members these matrices are nonsingular and are obtained from the submatrices of the matrix C(A) or C(B). Then we have

$$C(EQ) = \left[\begin{bmatrix} e^{-1} & 0 \\ 0 & z^{-1} \end{bmatrix} + \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{bmatrix} + \begin{bmatrix} \bar{b}_{11} & \bar{b}_{12} \\ \bar{b}_{21} & \bar{b}_{22} \end{bmatrix} \right]^{-1} \tag{2.1.4-7}$$

Forming the sum of the matrices as indicated, then simplifying

$$C(EQ) = \begin{bmatrix} e^{-1} + \bar{a}_{11} + \bar{b}_{11} & \bar{a}_{12} + \bar{b}_{12} \\ \bar{a}_{21} + \bar{b}_{21} & z^{-1} + \bar{a}_{22} + \bar{b}_{22} \end{bmatrix}^{-1} \tag{2.1.4-8}$$

$$C(EQ) = \begin{bmatrix} e^{-1} + c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}^{-1} \tag{2.1.4-8.1}$$

in which the c_{ij} are well defined. Using the results of section I3.9, we have

$$C(EQ) = \begin{bmatrix} Q^{-1} & -Q^{-1}Y \\ -XQ^{-1} & (c_{22}^{-1} + XQ^{-1}Y) \end{bmatrix} \tag{2.1.4-9}$$

in which X and Y are well defined combinations of the c_{ij}. Since, as we will show, $Q^{-1} \rightarrow 0$ as $\epsilon \rightarrow 0$, we have in the limit

$$C(EQ) = \begin{bmatrix} 0 & 0 \\ 0 & c_{22}^{-1} \end{bmatrix} \tag{2.1.4-10}$$

From equation I3.9-4.3 we have

$$Q = e^{-1} + c_{11} - c_{12}c_{22}^{-1}c_{21} = e^{-1} + g = \epsilon^{-1}I + g$$

where g is well defined. Thus we have

$$Q^{-1} = \epsilon I + h$$

and then

$$I = Q\,Q^{-1} = I + \epsilon^{-1}h + g\epsilon + gh,$$

therefore $0 = \epsilon^{-1}h + g\epsilon + gh.$

Let $\epsilon \rightarrow 0$, it is obvious that $h \rightarrow 0$ and thus $Q^{-1} \rightarrow 0$.

We need some rules to handle the computation of the inverse of the flexure matrix for ideal members.

1. In the inverse sum of the form

$$C(EQ) = \left(\sum C(j)^{-1}\right)^{-1} \qquad (2.1.4\text{-}11)$$

if any row (and column) of a $C(j)$ is zero, then the corresponding row (and column) of $C(EQ)$ is zero.

2. In calculating the inverse of a matrix having a row(s) and column(s) all zero we delete these row(s) and column(s) and compute the inverse of the remaining matrix. This is the z^{-1} in equation 2.1.4-5.

The situation could easily be more complicated than is indicated here. It is possible that a mamber may be idealized in a direction which is different from any of the three coordinate directions used to describe the system. If we make a transformation of the form

$$C = L^T C' L \qquad (2.1.4\text{-}12)$$

where L is a matrix of direction cosines, appropriately arranged(1). We can transform the C matrix so that the

direction of the idealization is one of the six principal directions. Highly complicated idealized members may require more sophisticated techniques in order to be usable in equations of the form of equation 2.1.4-11. However, it is the purpose of this section only to show how to handle simple idealizations.

2.1.5. Transfer Transformations - B Transformations.

Define the following matrices.

$$\text{Let} \quad b_U = \begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix} \qquad (2.1.5\text{-}1D)$$

where the elements of b_U are the coordinates of the point U. (The quantity b_U is sometimes called the tensor of the vector from the origin to the point U.[5])

$$\text{Let} \quad b_{UV} = b_U - b_V \qquad (2.1.5\text{-}2D)$$

$$\text{and} \quad B_{UV} = \begin{bmatrix} I & 0 \\ b_{UV} & I \end{bmatrix} \qquad (2.1.5\text{-}3D)$$

Then using equation I2.3-1 we have for the transpose of this matrix

$$B^T_{UV} = \begin{bmatrix} I & b^T_{UV} \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & -b_{UV} \\ 0 & I \end{bmatrix} \qquad (2.1.5\text{-}4)$$

2.1.5.1. Inverse of B-Transformation.

It is easy to verify that

$$(B_{UV})^{-1} = \begin{bmatrix} I & 0 \\ -b_{UV} & I \end{bmatrix} \qquad (2.1.5\text{-}5)$$

From equations 2.1.5-2D and 2.1.5-3D we have

$$(B_{UV})^{-1} = B_{VU} \qquad (2.1.5\text{-}6)$$

Also in a similiar manner

$$(B^T_{UV})^{-1} = B^T_{VU} \qquad (2.1.5\text{-}7)$$

2.1.5.2. Transitivity of B-Transformations.

From the definitions 2.1.5-2D and 2.1.5-3D

$$B_{UV}\, B_{VW} = \begin{bmatrix} I & 0 \\ b_{UV} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ b_{VW} & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ b_{UV} \quad b_{VW} & I \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ (b_U - b_V \quad b_V - b_W) & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ b_{UW} & I \end{bmatrix}$$

Thus $\quad B_{UV}\, B_{VW} = B_{UW} \qquad (2.1.5\text{-}8)$

By similiar reasoning and also using equation I3.4-1

$$B^T_{VW}\, B^T_{UV} = B^T_{UW} \qquad (2.1.5\text{-}9)$$

2.1.5.3. Identity B-Transformations.

In the process of simplifying some later equations we will have use for the following B transformation pairs. These are derived in the same manner as equations 2.1.5-8 and 2.1.5-9 (where we let $W = U$)

$$B_{UV}\, B_{VU} = I \qquad (2.1.5\text{-}10)$$

$$B_{VU}^{T} B_{UV}^{T} = I \qquad (2.1.5\text{-}11)$$

2.1.6. Application of the B-Transformation.

Using the definitions and relationships of the previous section and the application of elementary statics and kinematics, the following transformations can be derived.

$$F(J,U) = - B_{UV} F(J,V) \qquad (2.1.6\text{-}1D)$$

where both $F(J,U)$ and $F(J,V)$ are acting on the bar J. Also we can derive for the displacement transformation

$$D(U) = B_{VU}^{T} D(V)$$ if the member from U to V does not deform (2.1.6-2D)

These same transfer equations have been derived by others.(1), (2)

2.2. Discussion of a Simple Member.

Under the action of force $F(J,U)$, we have the displacement of the point U

$$D(U) = C(J,U)\ F(J,U) + B_{VU}^{T} D(V) \qquad (2.2\text{-}1D)$$

in which

$C(J,U)\ F(J,U)$ represents the deformation of member J due to force $F(J,U)$ acting on member J at U, end V being con-

sidered fixed; and

B^T_{VU} D(V) represents the transformation of the displacement of the point V, considering the member J as a rigid member. We may also think of this as the translation or projection of the displacement of the point V to the point U.

This is a fundamental equation of the thesis.

We consider the members of the structure to have two pertinent points. Usually the member will be attached to other members at these points. However, sometimes at one such point we will have only forces applied and displacements observed. The exact shape of the member is unimportant, as long as the flexure matrix for the member can be found. It may be computed from the geometric configuration of the beam,[(1), (3)] or it may be experimentally determined.

2.2.1. Equivalence of two Different Flexure Matrices Decribing the Same Member.

Apply a force F*(J,U), considering the end V constrained.

$$F^*(J,U) = - B_{UV}\ F(J,V)$$

and $D^*(U) = C(J,U)\ F^*(J,U) = - C(J,U)\ B_{UV}\ F(J,V)$

Apply a rigid body motion to the member J of amount -D*(U). This moves the point U to the position it would have had if U had been the constrained end of the member J.

$$D(V) = B^T_{UV}\ D^*(U) = B^T_{UV}\ C(J,U)\ B_{UV}\ F(J,V) \qquad (2.2.1\text{-}1)$$

Now separately apply a force $F(J,V)$, considering the end U constrained

$$D(V) = C(J,V)\ F(J,V) \qquad (2.2.1\text{-}2)$$

Since equations 2.2.1-1 and 2.2.1-2 are equal

$$C(J,V) = B_{UV}^{T}\ C(J,U)\ B_{UV} \qquad (2.2.1\text{-}3)$$

2.3. Discussion of a Simple Polyped.

A simple polyped is a structure, which may be part of a larger structure, having the following properties:

1. It is composed of individual members each having an associated flexure matrix C.

2. All the members of the polyped are joined at a common point, say P_i, called its nucleus. For simplicity we will designate a simple polyped by this nucleus, i.e., polyped P_i.

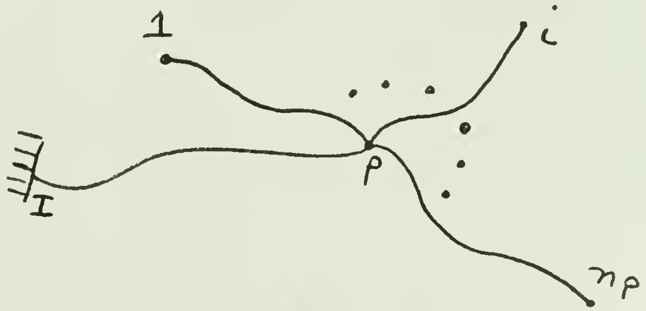
3. Forces are applied only at the ends of members, either the external end or the nucleus.

Members of the polyped are denoted by the designation of the external end of the member. (By external end we mean the end of the member not attached to the nucleus.) For example, the polyped member connecting nucleus P_i and external end j is denoted member j. Later in the thesis we will consider linked polypeds. When a member links two polyped nuclei, say polyped P_i and polyped P_j, the link member is denoted either as member P_j of polyped P_i, or as member P_i of polyped P_j, depending on whichever usage is more convenient at the moment.

2.3.1. A Simple Polyped with one Member Fixed.

Let us consider a simple polyped of which one member, say I, has its external end "fixed". By "fixed" is implied that the anchored or constrained end has only motions which are independent of the forces in the polyped. That is, any motions of the "fixed" end are known and are unrelated to the forces exerted within the polyped and its other members. In chapter 6 we will relax the constraints on the "fixed" points.

Let us assume no external force is applied to the nucleus P. Consider the following sketch for the ensuing development. In a sketch we will indicate only particular members, such as the fixed member and members linking to another polyped and a general member. There may be other members of the polyped, in fact in general there will be n_P external members in the P-th polyped.



From a consideration of the static equilibrium we have

$$F(I,P) + F(1,P) + F(2,P) + \dots + F(i,P) + \dots = 0$$

For convenience we will adopt the notation

$\sum_{i(P)} F(i,P)$ represents the summation of forces acting on each of the i external members of the

polyped P, excluding from the summation the fixed member. Later when we consider members linked at their external ends or when we link polypeds, the linking members are also excluded from the summation.

Then $$F(I,P) = -\sum_{i(P)} F(i,P) \qquad (2.3.1\text{-}1)$$

Using equation 2.1.6-1D we have

$$F(I,P) = \sum_{i(P)} B_{Pi}\, F(i,i) \qquad (2.3.1\text{-}2)$$

Now $D(P) = C(I,P)\, F(I,P)$

and substituting equation 2.3.1-2 we have

$$D(P) = C(I,P) \sum_{i(P)} B_{Pi}\, F(i,i) \qquad (2.3.1\text{-}3)$$

Then using the fundamental equation 2.2-1 we have for the displacement of the external end of a general member of the polyped P.

$$D(j) = C(j,j)\, F(j,j) + B^T_{Pj}\, D(P) \qquad (2.3.1\text{-}4)$$

Substituting equation 2.3.1-3

$$D(j) = C(j,j)\, F(j,j) + B^T_{Pj}\, C(I,P) \sum_{i(P)} B_{Pi}\, F(i,i)$$

(2.3.1-5)

In order to display this set of equations in a more compact and meaningful form let

$$D_P = \left\{ D(1), D(2), \ldots . D(j), \ldots . , D(n_P) \right\} \qquad (2.3.1\text{-}6D)$$

and $F_P = \left\{ F(1,1), F(2,2), \ldots\ldots, F(j,j), \ldots\ldots, F(n_P, n_P) \right\}$

(2.3.1-7D)

Note that each element in the column matrices D and F is itself a 6-element column matrix.

Then

$$D_P = \left[\left[\operatorname{diag}\left[C(i,i) \right] \right] + \left[B^T_{Pj}\ C(I,P)\ B_{Pi} \right] \right] F_P$$

(2.3.1-8)

Thus we have described the displacement of the external end of any member of the polyped due to forces on any combination of the external ends of the polyped in terms of the sum of relatively simple matrices.

The first term of this composite flexure matrix, namely $\operatorname{diag}\left[C(i,i)\right]$ enters only when the force on the i-th member is used in computing the i-th displacement. This $C(i,i)$ describes the flexure of the member from the nucleus to this i-th external end.

The second term enters for all forces and actually describes the contribution of each force to the elastic distortion of the member $\overline{IP}$. This elastic distortion is manifested in the displacement of the nucleus P. The displacement at the external end j is due to a rigid body translation of the displacement of the nucleus. Thus we see how the components which make up the composite flexure matrix enter into the general result.

2.3.2. Some Properties of $C^*(X,P)$.

Since the second term of the composite flexure matrix in equation 2.3.1-8 will be used repeatedly, let us use the notation

$$C^*_{ji}\,(X,P) = B^T_{Pj}\, C(X,P)\, B_{Pi} \qquad (2.3.2\text{-}1D)$$

Let us consider the tranpose of $C^*(X,P)$. Using equation I3.4-1 we have

$$\left(C^*_{ji}\,(X,P)\right)^T = \left(B^T_{Pj}\, C(X,P)\, B_{Pi}\right)^T = B^T_{Pi}\left(C(X,P)\right)^T (B^T_{Pj}\;)^T$$

Using equations 2.1.4-2 and I3.7-1 we have

$$\left(C^*_{ji}\,(X,P)\right)^T = B^T_{Pi}\, C(X,P)\, B_{Pj} = C^*_{ij}\,(X,P)$$

$$(2.3.2\text{-}2)$$

The C^* matrix is thus shown to be symmetric. This is as it must be for the polyped to obey the reciprocal theorem of elastic bodies.

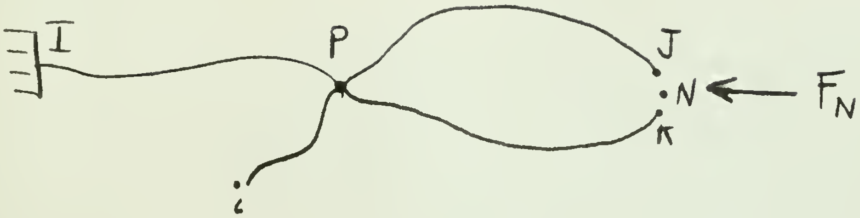
2.4. Simple Polyped with Multiple Connected Ends.

Let us consider first in detail having two members joined at a common external point. Then we will extend this to a third member connected at this point. The results can easily be generalized to n members joined at the same point.

2.4.1. Two Member Ends Connected Externally.

Again in this development we will assume the external end of member I is fixed and no "external" force acts at the nucleus of P. Consider that members J and K are joined at their external ends at point N. Consider the

following sketch.



From a consideration of static equilibrium at P we have

$$F(I,P) = B_{PJ}\ F(J,J) + B_{PK}\ F(K,K) + \sum_{i(P)} B_{Pi}\ F(i,i)$$

(2.4.1-1)

As before

$$D(P) = C(I,P)\ F(I,P)$$

After substituting equation 2.4.1-1

$$D(P) = C(I,P)\Big(B_{PJ}\ F(J,J) + B_{PK}\ F(K,K) + \sum_{i(P)} B_{Pi}\ F(i,i)\Big)$$

(2.4.1-2)

Now we will find the displacements $D(J)$ and $D(K)$ and set them equal to each other. Now we can eliminate both $F(J,J)$ and $F(K,K)$ and have the various $D(j)$'s in terms only of the forces on the external ends of members.

Using the fundamental equation 2.2-1 we have for the displacement of the external end of member J

$$D(J) = C(J,J)\ F(J,J) + B^T_{PJ}\ D(P)$$

Substituting equation 2.4.1-2 we have

$$D(J) = \Big(C(J,J) + B^T_{PJ}\ C(I,P)\ B_{PJ}\Big)\ F(J,J)$$

$$+ B^T_{PJ}\ C(I,P)\ B_{PK}\ F(K,K) + B^T_{PJ}\ C(I,P) \sum_{i(P)} B_{Pi}\ F(i,i)$$

(2.4.1-3)

Similiarly we have for the displacement of the external end of member K

$$D(K) = \left(C(K,K) + B_{PK}^{T}\ C(I,P)\ B_{PK}\right) F(K,K)$$
$$+ B_{PK}^{T}\ C(I,P)\ B_{PJ}\ F(J,J) + B_{PK}^{T}\ C(I,P) \sum_{i(P)} B_{Pi} F(i,i)$$

(2.4.1-4)

Now at the point N we have

$$F(J,J) + F(K,K) = F(N) \qquad (2.4.1\text{-}5)$$

Let us use the simplifying notation of equation 2.3.2-1D. Then substituting equations 2.3.2-1D and 2.4.1-5 in both equations 2.4.1-3 and 2.4.1-4 we have

$$D(J) = \left(C(J,J) + C^{*}_{JJ}\ (I,P) - C^{*}_{JK}\ (I,P)\right) F(J,J)$$
$$+ C^{*}_{JK}\ (I,P)\ F(N) + \sum_{i(P)} C^{*}_{Ji}\ (I,P)\ F(i,i)$$

(2.4.1-6)

and
$$D(K) = \left(- C(K,K) - C^{*}_{KK}\ (I,P) + C^{*}_{KJ}\ (I,P)\right)\ F(J,J)$$
$$+ \left(C(K,K) + C^{*}_{KK}\ (I,P)\right) F(N) + \sum_{i(P)} C^{*}_{Ki}\ (I,P)\ F(i,i)$$

(2.4.1-7)

Since the point N is the same point as the external ends of the members J and K we have

$$D(N) = D(J) = D(K) \qquad (2.4.1\text{-}8)$$

Equations 2.4.1-6 and 2.4.1-7 then become

$$D(N) = D(J) = C(J,J)\ F(J,J) + C^*_{NN}\ (I,P)\ F(N)$$

$$+ \sum_{i(P)} C^*_{Ni}\ (I,P)\ F(i,i) \qquad (2.4.1\text{-}9)$$

and $$D(N) = D(K) = -\ C(K,K)\ F(J,J) + \left(C(K,K) + C^*_{NN}\ (I,P)\right) F(N)$$

$$+ \sum_{i(P)} C^*_{Ni}\ (I,P)\ F(i,i) \qquad (2.4.1\text{-}10)$$

Since the C matrices are non-singular for ordinary members (for idealized members see section 2.1.4.1) we can solve the above equations and write

$$F(J,J) = \left(C(J,J) + C(K,K)\right)^{-1} C(K,K)\ F(N) \qquad (2.4.1\text{-}11)$$

Using equation I3.5-2, equation 2.4.1-11 becomes

$$F(J,J) = C(J,J)^{-1}\left(C(J,J)^{-1} + C(K,K)^{-1}\right)^{-1} F(N) \qquad (2.4.1\text{-}12)$$

Substituting in equation 2.4.1-5 and simplifying we have

$$F(K,K) = C(K,K)^{-1}\left(C(J,J)^{-1} + C(K,K)^{-1}\right)^{-1} F(N) \qquad (2.4.1\text{-}13)$$

Now let us find the various D(j) 's and D(N) as functions of the forces on various external ends of the members. First we must find D(P). From equation 2.4.1-2 and substituting equations 2.4.1-5 we have

$$D(P) = C(I,P)\left(\sum_{i(P)} B_{Pi}\ F(i,i) + B_{PN}\ F(N)\right) \qquad (2.4.1\text{-}14)$$

Applying the fundamental equation 2.2-1D we have for a typical external end of a member of polyped P

$$D(j) = C(j,j)\,F(j,j) + B^T_{Pj}\,C(I,P)\sum_{i(P)} B_{Pi}\,F(i,i) + B^T_{Pj}\,C(I,P)\,B_{PN}\,F(N) \qquad (2.4.1\text{-}15)$$

Substituting equation 2.4.1-12 in equation 2.4.1-6 we have for the displacement of the point N

$$D(N) = C(J,J)\left(C(J,J)+C(K,K)\right)^{-1} C(K,K)\,F(N) + B^T_{PN}\,C(I,P)\sum_{i(P)} B_{Pi}\,F(i,i) + B^T_{PN}\,C(I,P)\,B_{PN}\,F(N) \qquad (2.4.1\text{-}16)$$

The same results would be obtained if we substituted equation 2.4.1-12 in equation 2.4.1-7 and used also equation I4.2-1. Now substituting equation I4.1-1 in equation 2.4.1-16 we have

$$D(N) = \left(C(J,J)^{-1} + C(K,K)^{-1}\right)^{-1} F(N) + B^T_{PN}\,C(I,P)\sum_{i(P)} B_{Pi}\,F(i,i) + B^T_{PN}\,C(I,P)\,B_{PN}\,F(N) \qquad (2.4.1\text{-}17)$$

Now to exhibit the set of equations 2.4.1-15 and 2.4.1-17 in a form that displays the role of the various flexure C matrices, let

$$D_P = \left\{D(1),\ D(2), \ldots\ldots, D(j), \ldots\ldots, D(N)\right\} \qquad (2.4.1\text{-}18D)$$

$$F_P = \left\{F(1,1), F(2,2), \ldots\ldots, F(j,j), \ldots\ldots, F(N)\right\} \qquad (2.4.1\text{-}19D)$$

Then we have the following matrix equation

$$D_P = \left[\begin{bmatrix} \text{diag } C(j,j) & 0 \\ 0 & \left(C(J,J)^{-1} + C(K,K)^{-1}\right)^{-1} \end{bmatrix} + \begin{bmatrix} C^*_{ji}(I,P) \end{bmatrix} \right] F_P \qquad (2.4.1\text{-}20)$$

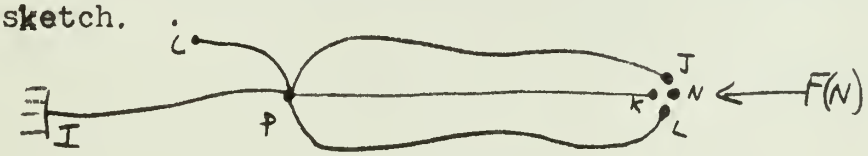
where as before

$$C^*_{ji}(I,P) = B^T_{Pj}\, C(I,P)\, B_{Pi} \qquad (2.3.2\text{-}1D)$$

Again we have described the displacement of the external ends of the polyped P due to forces on any combination of the external members of the polyped. The equivalent flexure matrix of the polyped is the sum of relatively simple matrices.

2.4.2. A Simple Polyped with more than two members joined at a common point.

Again in this development we will assume that the external end of member I is fixed and that no external forces act at the nucleus of P. Let us consider that three members are joined at a common external end. From this result and the results of the previous section it will be obvious how to generalize to n members joined at a common external point. Let us call these members J, K, and L. Consider the following sketch.



From a consideration of static equilibrium at P we have

$$F(I,P) = B_{PJ}\ F(J,J) + B_{PK}\ F(K,K) + B_{PL}\ F(L,L) + \sum_{i(P)} B_{Pi} F(i,i) \tag{2.4.2-1}$$

As before we have

$$D(P) = C(I,P)\left(B_{PJ}\ F(J,J) + B_{PK}\ F(K,K) + B_{PL}\ F(L,L) + \sum_{i(P)} B_{Pi}\ F(i,i)\right) \tag{2.4.2-2}$$

The method will be to find each of the displacements $D(J)$, $D(K)$, and $D(L)$ and set them equal to each other. Using the force equilibrium equation at point N we can determine the force on each member J, K, and L in terms of the external force at point N. Let us proceed to do this. Using the fundamental equation 2.2-1D, substituting equation 2.4.2-2, also simplifying the expression using equation 2.3.2-1D, then we have

$$D(J) = \left(C(J,J) + C^*_{JJ}\ (I,P)\right) F(J,J) + C^*_{JK}\ (I,P)\ F(K,K) + C^*_{JL}\ (I,P)\ F(L,L) + \sum_{i(P)} C^*_{Ji}\ (I,P)\ F(i,i) \tag{2.4.2-3}$$

and $$D(K) = C^*_{KJ}\ (I,P)\ F(J,J) + \left(C(K,K) + C^*_{KK}\ (I,P)\right) F(K,K) + C^*_{KL}\ (I,P)\ F(L,L) + \sum_{i(P)} C^*_{Ki}\ (I,P)\ F(i,i) \tag{2.4.2-4}$$

and $D(L) = C^*_{LJ}(I,P)\,F(J,J) + C^*_{LK}(I,P)\,F(K,K)$

$$+\Big(C(L,L) + C^*_{LL}(I,P)\Big)F(L,L) + \sum_{i(P)} C^*_{Li}(I,P)\,F(i,i) \qquad (2.4.2\text{-}5)$$

At the point N we have the following equation for the equilibrium of forces

$$F(J,J) + F(K,K) + F(L,L) = F(N) \qquad (2.4.2\text{-}6)$$

Solving equation 2.4.2-6 for F(K,K) and substituting in equation 2.4.2-3 and 2.4.2-4 and 2.4.2-5 we have

$$D(J) = \Big(C(J,J) + C^*_{JJ}(I,P) - C^*_{JK}(I,P)\Big)F(J,J)$$

$$+\Big(C^*_{JL}(I,P) - C^*_{JK}(I,P)\Big)F(L,L) + C^*_{JK}(I,P)\,F(N)$$

$$+\sum_{i(P)} C^*_{Ji}(I,P)\,F(i,i) \qquad (2.4.2\text{-}7)$$

and $D(K) = \Big(C^*_{KJ}(I,P) - C^*_{KK}(I,P) - C(KK)\Big)F(J,J)$

$$+\Big(C^*_{KL}(I,P) - C^*_{KK}(I,P) - C(K,K)\Big)F(L,L)$$

$$+\Big(C(K,K) + C^*_{KK}(I,P)\Big)F(N) + \sum_{i(P)} C^*_{Ki}(I,P)\,F(i,i) \qquad (2.4.2\text{-}8)$$

and $D(L) = \Big(C^*_{LJ}(I,P) - C^*_{LK}(I,P)\Big)F(J,J) + \Big(C(L,L)$

$$+ C^*_{LL}(I,P) - C^*_{LK}(I,P)\Big)F(L,L) + C^*_{LK}(I,P)\,F(N)$$

$$+\sum_{i(P)} C^*_{Li}(I,P)\,F(i,i) \qquad (2.4.2\text{-}9)$$

Since the point N is the same point as the external ends of the members J, K, and L we have

$$D(N) = D(J) = D(K) = D(L) \qquad (2.4.2\text{-}10)$$

The equations 2.4.2-7 through 2.4.2-9 become

$$D(N) = D(J) = C(J,J)\ F(J,J) + C^*_{NN}\ (I,P)\ F(N) + \sum_{i(P)} C^*_{Ni}\ (I,P)\ F(i,i) \qquad (2.4.2\text{-}11)$$

$$D(N) = D(K) = -\ C(K,K)\ F(J,J) - C(K,K)\ F(L,L) + \left(C(K,K) + C^*_{NN}\ (I,P)\right)F(N) + \sum_{i(P)} C^*_{Ni}\ (I,P)\ F(i,i) \qquad (2.4.2\text{-}12)$$

$$D(N) = D(L) = C(L,L)\ F(L,L) + C^*_{NN}\ (I,P)\ F(N) + \sum_{i(P)} C^*_{Ni}\ (I,P)\ F(i,i) \qquad (2.4.2\text{-}13)$$

Solving this set of equations we have

$$F(J,J) = C(J,J)^{-1}\left(C(J,J)^{-1} + C(K,K)^{-1} + C(L,L)^{-1}\right)^{-1} F(N) \qquad (2.4.2\text{-}14)$$

$$F(K,K) = C(K,K)^{-1}\left(C(J,J)^{-1} + C(K,K)^{-1} + C(L,L)^{-1}\right)^{-1} F(N) \qquad (2.4.2\text{-}15)$$

$$F(L,L) = C(L,L)^{-1}\left(C(J,J)^{-1} + C(K,K)^{-1} + C(L,L)^{-1}\right)^{-1} F(N) \qquad (2.4.2\text{-}16)$$

Note the similiarity between the above equations and equations 2.4.1-13 and 2.4.1-14. The extension to n members connected from the nucleus to the point N is obvious.

Now let us find the $D(j)$'s and the $D(N)$ as functions

of the various forces on the external ends of the members and the external force at N. The solution proceeds in the same manner as in section 2.4.1. As a matter of fact the displacement of the various members j is as before

$$D(j) = C(j,j)\ F(j,j) + \sum_{i(P)} C^{*}_{ji}\ (I,P)\ F(i,i) + C^{*}_{NN}(I,P)F(N)$$

(2.4.2-17)

The similiarity is expected since the displacement of the j-th end is not affected by how the point N is linked to the nucleus. The force F(N) affects this displacement only in the flexure of the member $\overline{IP}$.

In order to find D(N) we substitute equations 2.4.2-14 through 2.4.2-17 into any one of the equations 2.4.2-11 through 2.4.2-13. The result after simplifying is

$$D(N) = \left(C(J,J)^{-1} + C(K,K)^{-1} + C(L,L)^{-1}\right)^{-1} F(N)$$

$$+ \sum_{i(P)} C^{*}_{Ni}\ (I,P)\ F(i,i) + C^{*}_{NN}\ (I,P)\ F(N)$$

(2.4.2-18)

In order to exhibit the set of equations 2.4.2-17 and 2.4.2-18 in a simple and compact form we use as before equations 2.4.1-18D and 2.4.1-19D. The resulting matrix equation is thus

$$D_P = \left(\left[C_0\right] + \left[C^{*}_{ji}\ (I,P)\right]\right) F_P \qquad (2.4.2\text{-}19)$$

in which $\left[C_0\right]$ has the form

$$C_0 = \left[\begin{array}{c|c} \text{diag } C(j,j) & \begin{array}{c} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{array} \\ \hline 0\ 0\ \ldots\ 0 & \left(C(J,J)^{-1} + C(K,K)^{-1} + C(L,L)^{-1}\right)^{-1} \end{array}\right]$$

(2.4.2-19.1)

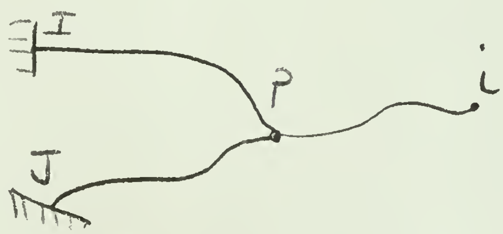
The extension to n members joined from the point N to the nucleus is only a change in the last term of the diagonal matrix C_0 which becomes the sum of the inverse of each member in this parallel-member arrangement, the sum being inverted.

Thus in equation 2.4.2-19 we have described the displacement of the external ends of the polyped P even if some members are joined at a common external point. If some other external point of the polyped also had several members joined there, then equation 2.4.2-19 is only complicated by having the equivalent flexibility of the parallel-member combination in the proper position of the diagonal matrix rather than a simple flexure matrix. We see here the beginning of a very general proposition -- namely, that a complex member network or substructure can be considered as a simple member if we use the appropriate equivalent flexibility matrix. However, we will develop this general proposition more fully later on.

2.4.3. A Simple Polyped with Two Members Fixed.

Let us consider the effect of having two members of the polyped with external ends fixed. We will derive an expres-

sion for the displacement of the nucleus P. This expression can then be used to find the displacement of the external ends using the method of the previous sections. Consider the following sketch.



We have for the displacement $D(P)$

$$D(P) = \dot{C}(I,P)\ F(I,P) \qquad (2.4.3\text{-}1)$$

and

$$D(P) = \dot{C}(J,P)\ F(J,P) \qquad (2.4.3\text{-}2)$$

From the consideration of force equilibrium at P we have

$$F(I,P) + F(J,P) = \sum_{i(P)} B_{Pi}\ F(i,i) \qquad (2.4.3\text{-}3)$$

Solving equations 2.4.3-1 and 2.4.3-2 and substituting into equation 2.4.3-3 we have

$$\left(I + C(J,P)^{-1}\ C(I,P)\right) F(I,P) = \sum_{i(P)} B_{Pi}\ F(i,i)$$

or

$$F(I,P) = \left(I + C(J,P)^{-1} C(I,P)\right)^{-1} \sum_{i(P)} B_{Pi}\ F(i,i) \qquad (2.4.3\text{-}4)$$

Substituting in equation 2.4.3-1 we have

$$D(P) = C(I,P)\left(I + C(J,P)^{-1} C(I,P)\right)^{-1} \sum_{i(P)} B_{Pi}\ F(i,i) \qquad (2.4.3\text{-}5)$$

Using equation I3.5-2 and simplifying we have

$$D(P) = \left(C(I,P)^{-1} + C(J,P)^{-1}\right)^{-1} \sum_{i(P)} B_{Pi}\, F(i,i) \qquad (2.4.3\text{-}6)$$

We now have an expression for $D(P)$ in terms only of the forces on the external ends of the members of the polyped. We can now use the methods of the previous sections to find the various displacements. Having more than one fixed member in the polyped causes no great complications, for the appropriate flexure matrix becomes the equivalent matrix for the parallel combination of members.

In the same manner that in sections 2.4.1 and 2.4.2 we generalized joining two members at their external ends to n members joined at their external ends, we may generalize here. The result is obvious and is the sum of the inverse of the individual flexure matrices of the fixed members, the resulting summation being inverted.

2.5. Simple Polyped with a Force Exerted at a Nucleus.

So far we have excluded having an external force exerted at a nucleus. Likewise we have excluded having the displacement of a nucleus appear in the final equations. The simplest way of handling such a case is to consider the force acting on the nucleus of the polyped as actually acting on a rigid member of infinitesimal length. The force thus becomes merely one of the several $F(i,i)$ and the terms of the matrices are as follows if this rigid member is the r-th member of the polyped:

1. the term in the diag $C(j,j)$ matrix, ie, $C(r,r)$ is a zero matrix.

2. the terms in the $C^*(I,P)$ matrix are

$$C^*_{jr}(I,P) = B^T_{Pj} C(I,P)\, B_{Pr}. \qquad (2.5\text{-}1)$$

However, since $r = P$, and $B_{Pr} = I$ we have

$$C^*_{jP}(I,P) = B^T_{Pj}\, C(I,P) \qquad (2.5\text{-}2)$$

This technique of associating a zero-length member with a force acting on a pertinent point (which is a nucleus) will be used whenever such forces are encountered. In this manner the final equations for each case will contain only pertinent points expressed as external ends of polypeds. No further mention of a force on a nucleus will be made.

In the same manner when we are interested in the displacement of a nucleus we will associate a zero-length rigid member and again the solution of a problem will be expressed in terms of pertinent points which are external ends of polypeds.

2.6. Generalizations of Notation.

Let us rewrite here some basic equations and definitions which will be used throughout the next chapters. We will also generalize the notation in anticipation of the requirements on notation in the following chapters. Also in the future specific reference to simplifications using the B-transformations will usually not be made.

Rewriting equation 2.2-1D

$$D(j) = C(P,j)\, F(P,j) + B^T_{Pj}\, D(P) \qquad (2.6\text{-}A)$$

Rewriting equation 2.3.2-1D, which expresses the translation of the flexure matrix within the same polyped, we have

$$C^*_{ji}\ (X,P) = B^T_{Pj}\ C(X,P)\ B_{Pi} \qquad (2.6\text{-B})$$

Let us adopt the more general notation which will be usable with multiple polypeds.

$C(P_i,P_j)$ = The flexure matrix of the member $\overline{P_iP_j}$ considered as if P_i were fixed and the force applied at P_j; and deflections measured at P_j. (2.6-C)

$F(P_i,P_j)$ = The force acting <u>on</u> member $\overline{P_iP_j}$ at P_j. Note that the second part of the parenthesis tells where the force acts. (2.6-D)

$\sum\limits_{i(P_j)} F(i,P_j)$ – The summation of the forces acting on each of the external members of the polyped P_j. The summation excludes members linking to other polypeds and to the fixed member. The forces act at the nucleus end of the members. (2.6-E)

Rewriting relation 2.6-E to indicate the forces acting at the external end of the members we have

$$\sum_{i(P_j)} F(i,P_j) = -\sum_{i(P_j)} F(P_j,i) \qquad (2.6\text{-F})$$

2.6.1. In section 2.3.2 we discussed a term of the composite flexure matrix which decribes the contribution of each force to the elastic distortion of a particular member, observed

as a displacement of the nucleus and then translated to the pertinent points of the polyped. Let us now generalize this and consider several polypeds. We are considering the contribution to the various translated displacements of each point in each of the polypeds due only to the elastic distortion in one particular member, say $\overline{P_xP_y}$. Considering only a force on the j-th member of the R-th polyped and observing the displacement at the external end of the i-th member of the S-th polyped, generalizing equation 2.3.2-1D, using notation similiar in form and meaning, we have

$$^{*}C_{R_j S_i}(P_x,P_y) = B^T_{P_y j}\, C(P_x,P_y)\, B_{P_y i} \qquad (2.6\text{-}G)$$

As it arises in the following chapters one must keep in mind to which polyped the i-th and j-th members belong. Hence tagging the $^{*}C$ with the polyped designation of the member will be of considerable assisstance. There are terms of the form of equation 2.6-G for the effect of each force individually on each displacement observed.

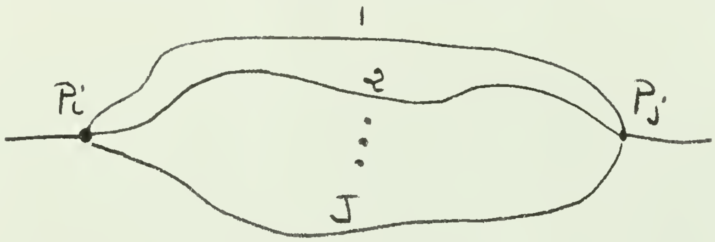
Then also for the matrix having equation 2.6-G as its general member we have for the general submatrix

$$^{*}C_{RS}(P_x,P_y) = \left[B^T_{P_y j}\, C(P_x,P_y)\, B_{P_y i} \right] \qquad (2.6\text{-}H)$$

where the j's are members of polyped R and the i's are members of the polyped S.

2.6.2. Equivalent flexure matrix of a parallel member combination.

In section 2.4.2 we derived an equivalent flexure matrix for the system of members shown in the sketch below.



Let us add a middle index to the flexure matrix notation to denote which of the several parallel members is being referred to, i.e., $C(P_i,J,P_j)$. The P_i and P_j have the same meaning as before, equation 2.6-C.

In order to shorten the expression for the equivalent flexure matrix of the combination of members and the show the generality of the parallel member combination let

$$\sum_J C(P_i,J,P_j)^{-1} = C(P_i,1,P_j)^{-1} + C(P_i,2,P_j)^{-1} + \ldots + C(P_i,J,P_j)^{-1} \qquad (2.6\text{-I})$$

The equivalent flexure matrix for the combination of members shown in the sketch is then

$$C(EQ) = \left(\sum_J C(P_i,J,P_j)^{-1}\right)^{-1} \qquad (2.6\text{-J})$$

CHAPTER 3

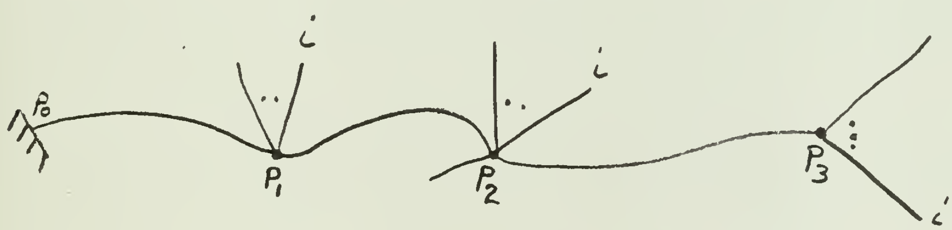
Chain Connected Polypeds

3. Discussion of Chain Connected Polypeds.

After the discussion of simple polypeds and the complications possible thereto, the next step in complexity appears to be to consider several polypeds linked one to another in a chain. The results obtained will allow us to consider this chain in the future either as a chain of simple polypeds or as a single complex polyped in building up more complicated structures. These polypeds may be linked either by having only a simple linkage from nucleus P_i to P_j or they may be linked by having several members linking from P_i to P_j. Let us consider these cases one at a time.

3.1. Simply Connected Chain of Polypeds.

Let us consider three polypeds linked as shown in the sketch below. From this result the generalization to more polypeds, each linked simply to the next one, will be obvious.



There may appear to be ambiguity about a member such as the one linking P_2 to P_3. Actually we may consider it either as an external end P_3 of polyped P_2 or as an external end P_2 of polyped P_3. We consider such a member to belong

to whichever polyped is more convenient.

From a consideration of the static equilibrium at point P_1 and using equation 2.6-F we have

$$F(P_0,P_1) = B_{P_1P_2}\, F(P_1,P_2) + \sum_{i(P_1)} B_{P_1 i}\, F(P_1,i) \qquad (3.1\text{-}1)$$

For the static equilibrium of point P_2 we have

$$F(P_1,P_2) = B_{P_2P_3}\, F(P_2,P_3) + \sum_{i(P_2)} B_{P_3 i}\, F(P_2,i) \qquad (3.1\text{-}2)$$

And for the point P_3 we have

$$F(P_2,P_3) = \sum_{i(P_3)} B_{P_3 i}\, F(P_3,i) \qquad (3.1\text{-}3)$$

As before, using equation 2.6-A we have

$$D(P_1) = C(P_0,P_1)\, F(P_0,P_1) \qquad (3.1\text{-}4)$$

Substituting equations 3.1-1 through 3.1-3 into equation 3.1-4 we have

$$D(P_1) = C(P_0,P_1)\left(\sum_{i(P_1)} B_{P_1 i}\, F(P_1,i) + \sum_{i(P_2)} B_{P_1 i}\, F(P_2,i) + \sum_{i(P_3)} B_{P_1 i}\, F(P_3,i)\right) \qquad (3.1\text{-}5)$$

Using equation 2.6-A and substituting equations 3.1-2 and 3.1-3, we have for $D(P_2)$

$$D(P_2) = C(P_1,P_2)\left(\sum_{i(P_2)} B_{P_2 i}\, F(P_2,i) + \sum_{i(P_3)} B_{P_2 i}\, F(P_3,i)\right)$$

$$+ B^T_{P_1 P_2}\, D(P_1) \qquad (3.1\text{-}6)$$

Similiarly using equation 2.6-A and substituting equation 3.1-3 we have

$$D(P_3) = C(P_2,P_3) \sum_{i(P_3)} B_{P_3 i}\, F(P_3,i) + B^T_{P_2 P_3}\, D(P_2) \qquad (3.1\text{-}7)$$

Now to find the displacement of the external ends of the members of the various polypeds we use equation 2.6-A and have

$$D_{P_1}(j) = C(P_1,j)\, F(P_1,j) + B^T_{P_1 j}\, C(P_0,P_1)\left(\sum_{i(P_1)} B_{P_1 i}\, F(P_1,i) + \sum_{i(P_2)} B_{P_1 i}\, F(P_2,i) + \sum_{i(P_3)} B_{P_1 i}\, F(P_3,i)\right) \qquad (3.1\text{-}8)$$

$$D_{P_2}(j) = C(P_2,j)\, F(P_2,j) + B^T_{P_1 i}\, C(P_0,P_1)\left(\sum_{i(P_1)} B_{P_1 i}\, F(P_1,i) + \sum_{i(P_2)} B_{P_1 i}\, F(P_2,i) + \sum_{i(P_3)} B_{P_1 i}\, F(P_3,i)\right)$$
$$+ B^T_{P_2 j}\, C(P_1,P_2)\left(\sum_{i(P_2)} B_{P_2 i}\, F(P_2,i) + \sum_{i(P_3)} B_{P_2 i}\, F(P_3,i)\right) \qquad (3.1\text{-}9)$$

and $$D_{P_3}(j) = C(P_3,j)\, F(P_3,j) + B^T_{P_1 j}\, C(P_0,P_1)\left(\sum_{i(P_1)} B_{P_1 i}\, F(P_1,i) + \sum_{i(P_2)} B_{P_1 i}\, F(P_2,i) + \sum_{i(P_3)} B_{P_1 i}\, F(P_3,i)\right) +$$

$$+ B^T_{P_2 j}\, C(P_1,P_2)\left(\sum_{i(P_2)} B_{P_2 i}\, F(P_2,i) + \sum_{i(P_3)} B_{P_2 i}\, F(P_3,i)\right)$$

$$+ B^T_{P_3 j}\, C(P_2,P_3) \sum_{i(P_3)} B_{P_3 i}\, F(P_3,i) \qquad (3.1\text{-}10)$$

In equation 3.1-8 j takes on all values from 1 to n_{P_1}, and in equation 3.1-9 j takes on all values from 1 to n_{P_2}, and in equation 3.1-10 j takes on all values from 1 to n_{P_3}. Thus we have a large system of linear equations when all these equations are considered simultaneously.

We can simplify and compact equations 3.1-8 through 3.1-10 if we use equation 2.6-G.

Then
$$D_{P_1}(j) = C(P_1,j)\, F(P_1,j) + \sum_{i(P_1)}{}^{*}C_{1_j 1_i}(P_0,P_1)\, F(P_1,i)$$
$$+ \sum_{i(P_2)}{}^{*}C_{1_j 2_i}(P_0,P_1)\, F(P_2,i) + \sum_{i(P_3)}{}^{*}C_{1_j 3_i}(P_0,P_1)\, F(P_3,i) \qquad (3.1\text{-}8.1)$$

and
$$D_{P_2}(j) = C(P_2,j)\, F(P_2,j) + \sum_{i(P_1)}{}^{*}C_{2_j 1_i}(P_0,P_1)\, F(P_1,i)$$
$$+ \sum_{i(P_2)}{}^{*}C_{2_j 2_i}(P_0,P_1)\, F(P_2,i) + \sum_{i(P_3)}{}^{*}C_{2_j 3_i}(P_0,P_1)\, F(P_3,i)$$
$$+ \sum_{i(P_2)}{}^{*}C_{2_j 2_i}(P_1,P_2)\, F(P_2,i) + \sum_{i(P_3)}{}^{*}C_{2_j 3_i}(P_1,P_2)\, F(P_3,i) \qquad (3.1\text{-}9.1)$$

and
$$D_{P_3}(j) = C(P_3,j)\, F(P_3,j) + \sum_{i(P_1)}{}^{*}C_{3_j 1_i}(P_0,P_1)\, F(P_1,i)$$
$$+ \sum_{i(P_2)}{}^{*}C_{3_j 2_i}(P_0,P_1)\, F(P_2,i) + \sum_{i(P_3)}{}^{*}C_{3_j 3_i}(P_0,P_1)\, F(P_3,i) +$$

$$+\sum_{i(P_2)}{}^{*}C_{3_j 2_i}(P_1,P_2)\ F(P_2,i)+\sum_{i(P_3)}{}^{*}C_{3_j 3_i}(P_1,P_2)\ F(P_3,i)$$

$$+\sum_{i(P_3)}{}^{*}C_{3_j 3_i}(P_2,P_3)\ F(P_3,i) \qquad (3.1\text{-}10.1)$$

It is obvious how the system of equations 3.1-8 through 3.1-10 extends to N polypeds. In order to display these equations in a more compact and meaningful form,

let $$D = \left\{ D_{P_1}, D_{P_2}, \ldots\ldots, D_{P_i}, \ldots, D_{P_N} \right\} \qquad (3.1\text{-}11D)$$

where each D_{P_i} is in turn a column matrix of the displacements of all the external members of the P_i polyped, ie,

$$D_{P_i} = \left\{ D(1), D(2), \ldots\ldots\ldots, D(j), \ldots\ldots D(n_{P_i}) \right\}$$

(3.1-12)

Also

$$F = \left\{ F(P_1,i), F(P_2,i), \ldots, F(P_j,i), \ldots, F(P_N,i) \right\}$$

(3.1-13D)

where each $F(P_j,i)$ is a column matrix of the forces on the external ends of the P_j polyped, ie,

$$F(P_j,i) = \left\{ F(P_j,1), F(P_j,2), \ldots\ldots, F(P_j,i), \ldots, F(P_j,n_{P_j}) \right\}$$

(3.1-14)

The generalization of equations 3.1-8 through 3.1-10 can thus be represented as

$$D = (C_0 + C_1 + C_2 + \ldots\ldots + C_N)\ F \qquad (3.1\text{-}15)$$

where each C_k is an N x N partitioned matrix. Using equation 2.6-H for each C_k we have

$$C_0 = \begin{bmatrix} \mathrm{diag}\left[C(P_1,i)\right] & 0 & \cdots & 0 \\ 0 & \mathrm{diag}\left[C(P_2,i)\right] & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \mathrm{diag}\left[C(P_N,i)\right] \end{bmatrix}$$

(3.1-16.0)

$$C_1 = \begin{bmatrix} {}^*C_{11}(P_0,P_1) & {}^*C_{12}(P_0,P_1) & \cdots & {}^*C_{1N}(P_0,P_1) \\ {}^*C_{21}(P_0,P_1) & {}^*C_{22}(P_0,P_1) & \cdots & {}^*C_{2N}(P_0,P_1) \\ \cdots & \cdots & \cdots & \cdots \\ {}^*C_{N1}(P_0,P_1) & {}^*C_{N2}(P_0,P_1) & \cdots & {}^*C_{NN}(P_0,P_1) \end{bmatrix}$$

(3.1-16.1)

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & {}^*C_{22}(P_1,P_2) & {}^*C_{23}(P_1,P_2) & \cdots & {}^*C_{2N}(P_1,P_2) \\ 0 & {}^*C_{32}(P_1,P_2) & {}^*C_{33}(P_1,P_2) & \cdots & {}^*C_{3N}(P_1,P_2) \\ & \cdots & \cdots & \cdots & \cdots \\ 0 & {}^*C_{N2}(P_1,P_2) & {}^*C_{N3}(P_1,P_2) & \cdots & {}^*C_{NN}(P_1,P_2) \end{bmatrix}$$

(3.1-16.2)

$$C_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & {}^*C_{33}(P_2,P_3) & {}^*C_{34}(P_2,P_3) & \cdots & {}^*C_{3N}(P_2,P_3) \\ 0 & 0 & {}^*C_{43}(P_2,P_3) & {}^*C_{44}(P_2,P_3) & \cdots & {}^*C_{4N}(P_2,P_3) \\ \cdots & & \cdots & \cdots & \cdots\cdot & \cdots \\ 0 & 0 & {}^*C_{N3}(P_2,P_3) & {}^*C_{N4}(P_2,P_3) & \cdots & {}^*C_{NN}(P_2,P_3) \end{bmatrix}$$

(3.1-16.3)

. . .

$$C_k = \begin{bmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & {}^*C_{kk}(P_{k-1},P_k) & \cdots & {}^*C_{kN}(P_{k-1},P_k) \\ 0 & 0 & \cdots & {}^*C_{k\quad k}(P_{k-1},P_k) & \cdots & {}^*C_{k\quad N}(P_{k-1},P_k) \\ \cdots & \cdots & & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & {}^*C_{Nk}(P_{k-1},P_k) & \cdots & {}^*C_{NN}(P_{k-1},P_k) \end{bmatrix}$$

(3.1-16.k)

$$C_N = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & & \cdots & \cdots \\ 0 & 0 & \cdots & {}^*C_{NN}(P_{N-1},P_N) \end{bmatrix}$$

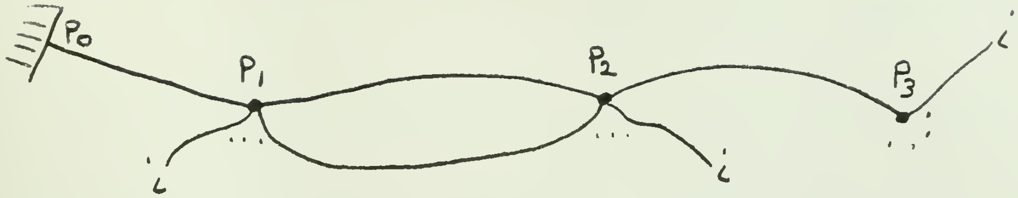
(3.1-16.N)

Each of the C_k matrices is partitioned into N rows and N columns of partition segments. (The term partition segment is coined for convenience in exposition-it is the first partitioning of a matrix such as C_k). Each of the partition segments is itself a matrix. Thus the RS-th segment, that is the segment in the R-th row and S-th column, is a matrix each element of which is a 6 x 6 matrix. The RS-th segment is an array of matrices n_{P_S} columns by n_{P_R} rows. (There are n_{P_i} external ends to the P_i polyped).

Thus we have described the displacement of any external end of this simple polyped chain for any combination of forces at the external ends of members of the complex structure. The equivalent flexure matrix is the sum of the C_i flexure matrix terms. We see that this chain of simple polypeds may also be considered as a complex polyped having more than one nucleus, namely nuclei P_1, P_2, ..., P_N. The sum of the C_k matrix terms is then the flexure matrix for the complex polyped. Since our aim is to be able to consider very complicated structures without having to start afresh with simple members each time, we now have a powerful tool for considering more complicated structures.

3.2. Simple Chain of Polypeds with a Multiple Linkage between Polypeds.

In this case we have several linkages between the polyped P_i and the polyped P_j. We may consider the linkages as having a certain flexure matrix defined over the length of the member. Consider the following sketch.



We do not need to start completely afresh; we may take some previous results, restate the equations in more general notation, and extend the concepts from there. Using equation 2.4.2-18, if we let point N be P_2 and write for F(N)

$$F(N) = B_{P_2P_3}\, F(P_2,P_3) + \sum_{i(P_2)} B_{P_2 i}\, F(P_2,i) \tag{3.2-1}$$

and if we generalize $C^*_{Ni}\ (I,P)$ using section 2.6.1 we have for $D(P_2)$

$$\begin{aligned} D(P_2) = &\left[\sum_J C(P_1,J,P_2)^{-1}\right]^{-1}\left(B_{P_2P_3}F(P_2,P_3) + \sum_{i(P_2)} B_{P_2 i}F(P_2,i)\right) \\ &+ \sum_{i(P_1)} B^T_{P_1P_2}\, C(P_0,P_1)\, B_{P_1 i}\, F(P_1,i) \\ &+ B^T_{P_1P_2} C(P_0,P_1)\left(B_{P_2P_3}F(P_2,P_3) + \sum_{i(P_2)} B_{P_2 i}F(P_2,i)\right) \end{aligned} \tag{3.2-2}$$

Using equation 3.1-3 for $F(P_2,P_3)$, and using equation 2.6-A, substituting 3.2-2 we have

$$\begin{aligned} &D(P_3) + C(P_2,P_3) \quad B_{P_3 i}\, F(P_3,i) \\ &+ B^T_{P_2P_3}\left[\sum_J C(P_1,J,P_2)^{-1}\right]^{-1}\left(\sum_{i(P_2)} B_{P_2 i}F(P_2,i) + \sum_{i(P_3)} B_{P_3 i}F(P_3,i)\right) \end{aligned}$$

$$+B^T_{P_1P_3} \quad C(P_0,P_1)\left(\sum_{i(P_1)} B_{P_1 i} F(P_1,i) + \sum_{i(P_2)} B_{P_2 i} F(P_2,i) \right.$$

$$\left. + \sum_{i(P_3)} B_{P_3 i} \; F(P_3,i) \right) \qquad (3.2\text{-}3)$$

In order to bring out the extension to N polypeds more clearly let us use the following type of summation.

Let
$$\sum_{m=P_1}^{P_N} \sum_{i(m)} A_{m\,i} \; F(m,i) = \sum_{i(P_1)} A_{P_1 i} \; F(P_1,i) + \sum_{i(P_2)} A_{P_2 i} \; F(P_2,i) + \cdots + \sum_{i(P_N)} A_{P_N i} \; F(P_N,i) \qquad (3.2\text{-}4D)$$

We have from the previous section equation 3.1-8.1 for $D_{P_1}(j)$. Using equation 3.2-4D it becomes

$$D_{P_1}(j) = C(P_1,j)\; F(P_1,j) + \sum_{m=P_1}^{P_3} \sum_{i(m)} {}^{*}C_{1_j m_i}(P_0,P_1)\; F(m,i) \qquad (3.2\text{-}5)$$

Now to find the displacement of the external ends of the members of the polypeds P_2 and P_3 use equation 2.6-A and simplify using equation 2.6-G. We have

$$D_{P_2}(j) = C(P_2,j)\; F(P_2,j) + \sum_{m=P_1}^{P_3} \sum_{i(m)} {}^{*}C_{2_j m_i}(P_0,P_1)\; F(m,i)$$
$$+ B^T_{P_1 j} \left[\sum_{J} C(P_1,J,P_2)^{-1} \right]^{-1} \sum_{m=P_2}^{P_3} \sum_{i(m)} B_{mi} \; F(m,i) \qquad (3.2\text{-}6)$$

$$\text{and}\quad D_{P_3}(j) = C(P_3,j)\, F(P_3,j) + \sum_{m=P_1}^{P_3} \sum_{i(m)}{}^{*}C_{3_{j}m_i}(P_0,P_1)\, F(m,i)$$

$$+ B^T_{P_1 j}\left[\sum_J C(P_1,J,P_2)^{-1}\right]^{-1} \sum_{m=P_2}^{P_3} \sum_{i(m)} B_{mi}\, F(m,i)$$

$$+ \sum_{m=P_3} \sum_{i(m)}{}^{*}C_{3_{j}m_i}(P_2,P_3)\, F(m,i) \qquad (3.2\text{-}7)$$

As before we can use equations 3.1-11D and 3.1-13D to represent the system in the form

$$D = \bar{C}\, F \quad ,\text{where}$$

$$\bar{C} = C_0 + C_1 + C_2 + C_3 \qquad (3.2\text{-}8)$$

This equation is quite similiar to equation 3.1-15. Now if we enumerate the C_k in the same manner as in equations 3.1-16.k we find that the C_0, C_1, and C_3 are the same as the previous case. In C_2 we find that the equivalent flexure of a parallel combination of members, equation 2.6-J, has replaced the simple flexure matrix of the linkage from P_1 to P_2.

Thus a chain of polypeds, whether complicated by parallel members linking polypeds or not, may be represented by equations 3.1-15 and 3.1-16. In the case of multiple members linking polypeds we use the equivalent flexure matrix of the combination of members in the appropriate equation 3.1-16.k.

CHAPTER 4

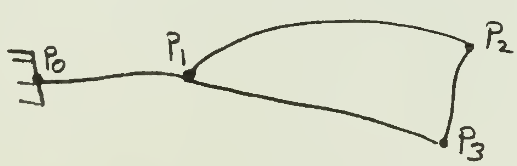
Loop Connected Polypeds

4. Discussion of Loop Connected Polypeds.

In the previous chapter we discussed multiple polypeds in which there is only one set of linkages from any one end of a chain of polypeds to the other ends. There may have been multiple members linking two polyped nuclei, but there is only one "path" or sequence of polyped nuclei between any two polypeds of the chain of polypeds. Let us consider the situation in which there are two distinct sets of linkages between two polypeds, that is we can proceed from a certain polyped, say P_i, to another polyped, say P_j, along a "path" of polyped nuclei which includes polyped P_a or along another "path" of polyped nuclei which does not include polyped nucleus P_a. We will refer to such a case as "loop connected". In analyzing this loop of polypeds we need only consider simple linkage between any two polypeds, for if it were a multiple linkage we could replace it with an equivalent simple linkage having a flexure matrix as indicated in equation 2.6-J. We will consider first a loop of three polypeds and then a loop of four polypeds which can be generalized to N polypeds in a loop.

4.1. Three Polyped Loop.

Let us consider three polypeds linked as shown in the sketch below. Only the linkage members are shown.



From a consideration of the static equilibrium at point P_1 we have

$$F(P_0,P_1) = B_{P_1P_2}F(P_1,P_2) + B_{P_1P_3}F(P_1,P_3) + \sum_{i(P_1)} B_{P_1i}F(P_1,i) \quad (4.1\text{-}1)$$

Similiarly at point P_2 we have

$$F(P_1,P_2) = B_{P_2P_3}F(P_2,P_3) + \sum_{i(P_2)} B_{P_2i}F(P_2,i) \quad (4.1\text{-}2)$$

and at point P_3 we have

$$F(P_1,P_3) + F(P_2,P_3) = \sum_{i(P_3)} B_{P_3i}F(P_3,i) \quad (4.1\text{-}3)$$

As before, using equation 2.6-A we have

$$D(P_1) = C(P_0,P_1)\ F(P_0,P_1) \quad (4.1\text{-}4)$$

Substituting equations 4.1-1 through 4.1-3 into equation 4.1-4 we have

$$D(P_1) = C(P_0,P_1)\Big(\sum_{i(P_1)} B_{P_1i}\ F(P_1,i) + \sum_{i(P_2)} B_{P_1i}\ F(P_2,i) + \sum_{i(P_3} B_{P_1i}\ F(P_3,i)\Big) \quad (4.1\text{-}5)$$

or expressed as a summation

$$D(P_1) = C(P_0,P_1) + \sum_{m=P_1}^{P_3} \sum_{i(m)} B_{P_1i}\ F(m,i) \quad (4.1\text{-}6)$$

We will find the displacement of $D(P_3)$ in two ways. First as a chain of polypeds $P_1P_2P_3$, and second as a chain

of polypeds P_1P_3. Setting these two expressions equal we will be able to express the forces in the linking members and the displacements of the various nuclei as functions only of the external forces on the member ends.

Using equation 2.6-A we have

$$D(P_2) = C(P_1,P_2)\ F(P_1,P_2) + B^T_{P_1P_2}\ D(P_1) \tag{4.1-6}$$

Now substituting equation 4.1-2 we have

$$D(P_2) = C(P_1,P_2)\left(B_{P_2P_3}\ F(B_2,P_3) + \sum_{i(P_2)} B_{P_2i}\ F(P_2,i)\right) + B^T_{P_1P_2}\ D(P_1) \tag{4.1-6.1}$$

For the moment there is no point in substituting for $D(P_1)$. Now using equation 2.6-A to obtain $D(P_3)$ from $D(P_2)$ we have

$$D(P_3) = \left(C(P_2,P_3) + B^T_{P_2P_3}\ C(P_1,P_2)\ B_{P_2P_3}\right)F(P_2,P_3) + \sum_{i(P_2)} B^T_{P_2P_3}\ C(P_1,P_2)\ B_{P_2i}\ F(P_2,i) + B^T_{P_1P_3}\ D(P_1) \tag{4.1-7}$$

Using equation 2.6-A to obtain $D(P_3)$ directly from $D(P_1)$, after substituting equation 4.1-3 we have

$$D(P_3) = -\ C(P_1,P_3)\ F(P_2,P_3) + C(P_1,P_3)\sum_{i(P_3)} B_{P_3i}\ F(P_3,i) + B^T_{P_1P_3}\ D(P_1) \tag{4.1-8}$$

Now equating 4.1-7 and 4.1-8 we have

$$\left(C(P_1,P_3) + C(P_2,P_3) + B^T_{P_2P_3}\ C(P_1,P_2)\ B_{P_2P_3}\right)F(P_2,P_3)$$

$$= C(P_1,P_3)\sum_{\iota(P_3)} B_{P_3i}\ F(P_3,i) \ - B^T_{P_2P_3}\ C(P_1,P_2)\sum_{\iota(P_2)} B_{P_2i}F(P_2,i)$$

(4.1-9)

Using equations I5.2-1D and I5.4-1D and solving equation 4.1-9 for $F(P_2,P_3)$ we have

$$F(P_2,P_3) = \left((\Sigma C)_{P_3}\right)^{-1} C(P_1,P_3) \sum_{\iota(P_3)} B_{P_3i}\ F(P_3,i)$$

$$- \left((\Sigma C)_{P_3}\right)^{-1} C(P_1,P_2)_{P_3} \sum_{\iota(P_2)} B_{P_3i}\ F(P_2,i)$$

(4.1-10)

where in this situation equation I5.4-1D becomes

$$(\Sigma C)_{P_3} = C(P_1,P_3) + C(P_1,P_2)_{P_3} + C(P_2,P_3)$$

Substituting equation 4.1-10 into equation 4.1-7 or 4.1-8 and simplifying using an appropriate equation from section I4, we have

$$D(P_3) = C(P_1,P_3)\ \left((\Sigma C)_{P_3}\right)^{-1} C(P_1,P_2)_{P_3}\sum_{\iota(P_2)} B_{P_3i}\ F(P_2,i)$$

$$+ C(P_1,P_3)\left((\Sigma C)_{P_3}\right)^{-1}\left(C(P_1,P_2)_{P_3} + C(P_2,P_3)\right)\cdot$$

$$\cdot\sum_{\iota(P_3)} B_{P_3i}\ F(P_3,i) + C(P_0,P_1)\cdot\sum_{m=P_1}^{P_3}\sum_{\iota(m)} B_{P_1i}\ F(m,i)$$

(4.1-11)

We can simplify equation 4.1-11 using equations I4.3-1 and I4.1-1.

$$D(P_3) = \Big[C(P_1,P_3)^{-1} + C(P_1,P_2)_{P_3}{}^{-1} \cdot C(P_2,P_3) \cdot C(P_1,P_3)^{-1}$$

$$+ C(P_1,P_2)_{P_3}{}^{-1} \Big]^{-1} \sum_{i(P_2)} B_{P_3 i}\; F(P_2,i)$$

$$+ \Big[C(P_1,P_3)^{-1} + \big(C(P_1,P_2)_{P_3} + C(P_2,P_3) \big)^{-1} \Big]^{-1} \sum_{i(P_3)} B_{P_3 i} F(P_3,i)$$

$$+ B^T_{P_1 P_3}\; C(P_0,P_1) \sum_{m=P_1}^{P_3} \sum_{i(m)} B_{P_1 i}\; F(m,i) \qquad (4.1\text{-}11.1)$$

Now substituting equation 4.1-10 into equation 4.1-2 we have

$$F(P_1,P_2) = B_{P_2 P_3} \left((\Sigma C)_{P_3} \right)^{-1} C(P_1,P_3) \sum_{i(P_3)} B_{P_3 i}\; F(P_3,i)$$

$$\Big[I - B_{P_2 P_3} \left((\Sigma C)_{P_3} \right)^{-1} C(P_1,P_2)_{P_3} \cdot B_{P_3 P_2} \Big] \sum_{i(P_2)} B_{P\; i} F(P_2,i)$$

$$(4.1\text{-}12)$$

This equation may be written in the following form in order to simplify it.

$$F(P_1,P_2) = B_{P_2 P_3} \left((\Sigma C)_{P_3} \right)^{-1} B^T_{P_2 P_3} \cdot B^T_{P_3 P_2}\; C(P_1,P_3)\; B_{P_3 P_2} \cdot$$

$$\cdot \sum_{i(P_3)} B_{P_2 i}\; F(P_3,i) + \Big[I - B_{P_2 P_3} \left((\Sigma C)_{P_3} \right)^{-1} B^T_{P_2 P_3} \cdot$$

$$\cdot B^T_{P_3 P_2}\; C(P_1,P_2)_{P_3}\; B_{P_3 P_2} \sum_{i(P_2)} B_{P_2 i}\; F(P_2,i)$$

$$(4.1\text{-}12.1)$$

Using equations I5.2-6 and I5.4-7 to simplify equation 4.1-12.1

$$F(P_1,P_2) = \left((\Sigma C)_{P_2}\right)^{-1} C(P_1,P_3)_{P_2} \sum_{i(P_3)} B_{P_2 i} F(P_3,i)$$

$$+\left[I - (C)_{P_2}^{-1} C(P_1,P_2)\right] \sum_{i(P_2)} B_{P_2 i} F(P_2,i)$$

(4.1-13)

Substituting equation 4.1-13 into equation 4.1-6 and simplifying using equation I4.2-1 we have

$$D(P_2) = C(P_1,P_2)\left((\Sigma C)_{P_2}\right)^{-1}\left(C(P_1,P_3)_{P_2} + C(P_3,P_2)\right) \cdot$$

$$\cdot \sum_{i(P_2)} B_{P_2 i} F(P_2,i)$$

$$+ C(P_1,P_2)\left((\Sigma C)_{P_2}\right)^{-1} C(P_1,P_3)_{P_2} \sum_{i(P_3)} B_{P_3 i} F(P_3,i)$$

$$+ B^T_{P_1 P_2} C(P_0,P_1) \sum_{m=P_1}^{P_3} \sum_{i(m)} B_{P_1 i} F(m,i)$$

(4.1-14)

We can again use the simplifying equations I4.3-1 and I4.2-2.

$$D(P_2) = \left[C(P_1,P_2)^{-1} \cdot \left(C(P_1,P_3)_{P_2} + C(P_3,P_2)\right)^{-1}\right]^{-1} \cdot$$

$$\cdot \sum_{i(P_2)} B_{P_2 i} F(P_2,i) + \left[C(P_1,P_2)^{-1}\right.$$

$$\left. + C(P_1,P_3)_{P_2}^{-1} C(P_3,P_2)\, C(P_1,P_2)^{-1} + C(P_1,P_3)_{P_2}^{-1}\right]^{-1} \cdot$$

$$\cdot \sum_{i(P_3)} B_{P_2 i} F(P_3,i) + B^T_{P_1 P_2}\ C(P_0,P_1) \sum_{m=P_1}^{P_3} \sum_{i(m)} B_{P_1 i}\ F(m,i)$$

(4.1-14.1)

Starting with equations 4.1-5.1, 4.1-14, and 4.1-11 and using equation 2.6-A we can find the displacement of the external ends of the members of the various polypeds. Let us simplify the various equations using equation 2.6-G. Then we have

$$D_{P_1}(j) = C(P_1,j)\ F(P_1,j) + \sum_{m=P_1}^{P_3} \sum_{i(m)}{}^{*} C_{1_j m_i}(P_0,P_1)\ F(m,i)$$

(4.1-15)

and
$$D_{P_2}(j) = C(P_2,j)\ F(P_2,j) + \sum_{m=P_1}^{P_3} \sum_{i(m)}{}^{*} C_{2_j m_i}(P_0,P_1)\ F(m,i)$$

$$+ B^T_{P_2 j}\left[C(P_1,P_2)\left((\Sigma C)_{P_2}\right)^{-1}\left(C(P_1,P_3)_{P_2} + C(P_3,P_2)\right)\right] \cdot$$

$$\cdot \sum_{i(P_2)} B_{P_2 i}\ F(P_2,i)$$

$$+ B^T_{P_2 j}\left[C(P_1,P_2)\left((\Sigma C)_{P_2}\right)^{-1}\ C(P_1,P_3)_{P_3}\right] \sum_{i(P_3)} B_{P_2 i} F(P_3,i)$$

(4.1-16)

and
$$D_{P_3}(j) = C(P_3,j)\ F(P_3,j) + \sum_{m=P_1}^{P_3} \sum_{i(m)}{}^{*} C_{3_j m_i}(P_0,P_1)\ F(m,i)$$

$$+ B^T_{P_3 j}\left[C(P_1,P_3)\left((\Sigma C)_{P_3}\right)^{-1}\ C(P_1,P_2)_{P_3}\right] \sum_{i(P_2)} B_{P_3 i} F(P_2,i)$$

$$+ B^T_{P_3 j}\left[C(P_1,P_3)\left((\Sigma C)_{P_3}\right)^{-1}\left(C(P_1,P_2)_{P_3} + C(P_2,P_3)\right)\right] \cdot$$

$$\cdot \sum_{i(P_3)} B_{P_3 i}\ F(P_3,i) \qquad (4.1\text{-}17)$$

In order to display these sets of equations in a more compact and meaningful form, as before, use equations 3.1-11D and 3.1-13D to define two column matrices; a system displacement D, and a system force F. Then we can display equations 4.1-15 through 4.1-17 in the form

$$D = (C_0 + C_1 + G)\ F \qquad (4.1\text{-}18)$$

in which C_0 and C_1 are as before equations 3.1-16.0 and 3.1-16.1 . The G is a flexure matrix in which the partition segments are rather complex combinations of individual member C matrices.

$$G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & G_{22} & G_{23} \\ 0 & G_{32} & G_{33} \end{bmatrix} \qquad (4.1\text{-}19)$$

where the general term of each G partition segment is

$$G_{2_j 2_i} = B^T_{P_2 j}\ C(P_1,P_2)\left((\Sigma C)_{P_2}\right)^{-1}\left(C(P_1,P_3)_{P_2} + C(P_3,P_2)\right) B_{P_2 i}$$

(4.1-19.22)

$$G_{2_j 3_i} = B^T_{P_2 j}\ C(P_1,P_2)\left((\Sigma C)_{P_2}\right)^{-1} C(P_1,P_3)_{P_2}\ B_{P_2 i}$$

(4.1-19.23)

$$G_{3_j 2_i} = B^T_{P_3 j}\ C(P_1,P_3)\left((\Sigma C)_{P_3}\right)^{-1} C(P_1,P_2)_{P_3}\ B_{P_3 i}$$

(4.1-19.32)

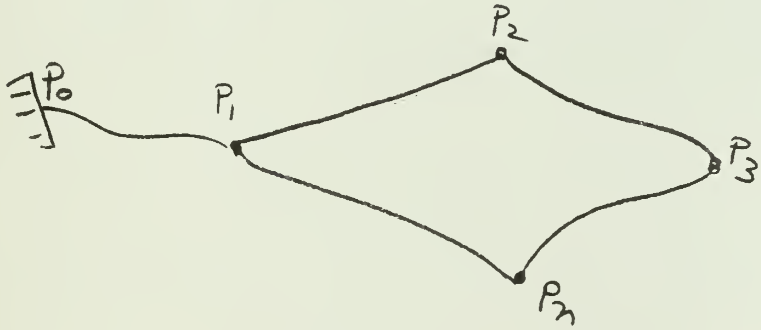
$$G_{3_j 3_i} = B^T_{P_3 j}\ C(P_1,P_3)\left((\Sigma C)_{P_3}\right)^{-1}\left(C(P_1,P_2)_{P_3} + C(P_2,P_3)\right) B_{P_3 i}$$

(4.1-19.33)

Now it can be shown that the G matrix is symmetric; however, let us defer doing this until we have the general case and do it once for all cases. In the general case the character of the G_{RS} terms will be more apparent. The main purpose of this section has been to follow through the system of analysis on a situation which is not too complicated. As can be seen from the equations 4.1-19.RS the situation is a complicated combination of the individual member flexure matrices at best. However, the structure we are analyzing is also very complicated and the analysis has yielded a set of linear equations to describe all the external displacements as functions of the various forces applied to the structure.

4.2. Four Polyped Loop.

Let us consider the four polyped loop linked as shown in the following sketch.



From a consideration of the static equilibrium at points P_1, P_2, P_3, and P_n we have

$$F(P_0,P_1) = B_{P_1P_n}F(P_1,P_n) + B_{P_1P_2}F(P_1,P_2) + \sum_{i(P_1)} B_{P_1i}\, F(P_1,i)$$

(4.2-1)

$$F(P_1,P_2) = B_{P_2P_3}F(P_2,P_3) + \sum_{i(P_2)} B_{P_2 i}\ F(P_2,i) \tag{4.2-2}$$

$$F(P_2,P_3) = B_{P_3P_n}F(P_3,P_n) + \sum_{i(P_3)} B_{P_3 i}\ F(P_3,i) \tag{4.2-3}$$

$$F(P_1,P_n) + F(P_3,P_n) = \sum_{i(P_n)} B_{P_n i}\ F(P_n,i) \tag{4.2-4}$$

As before using equation 2.6-A and substituting equations 4.2-1 through 4.2-4 we have

$$D(P_1) = C(P_0,P_1)\sum_{m=P_1}^{P_n}\sum_{i(m)} B_{P_1 i}\ F(m,i) \tag{4.2-5}$$

We will find the displacement of $D(P_n)$ in two way, first as a chain of polypeds $P_1\ P_2\ P_3\ P_n$, and secondly as a chain of polypeds $P_1\ P_n$. Setting these two expressions equal, after simplifying we will be able to express the forces in the linking members as functions of the external forces only.

As in the previous section we have

$$D(P_2) = C(P_1,P_2)\ F(P_1,P_2) + B^T_{P_1P_2}\ D(P_1) \tag{4.2-6}$$

$$D(P_3) = \left(C(P_2,P_3) + B^T_{P_2P_3}\ C(P_1,P_2)\ B_{P_2P_3}\right)F(P_2,P_3)$$

$$+ B^T_{P_2P_3}\ C(P_1,P_2)\sum_{i(P_2)} B_{P_2 i}\ F(P_2,i) + B^T_{P_1P_3}\ D(P_1) \tag{4.2-7}$$

Using a simplification of notation (equation I5.2-1D) we have

$$D(P_3) = \left(C(P_2,P_3) + C(P_1,P_2)_{P_3}\right)F(P_2,P_3)$$

$$+ C(P_1,P_2)_{P_3} \sum_{i(P_2)} B_{P_3 i} F(P_2,i) + B^T_{P_1P_3} D(P_1)$$

(4.2-7.1)

Again using equation 2.6-A and substituting equation 4.2-4 we have

$$D(P_n) = \left(C(P_3,P_n) + C(P_2,P_3)_{P_n} + C(P_1,P_2)_{P_n}\right)F(P_3,P_n)$$

$$+\left(C(P_2,P_3)_{P_n} + C(P_1,P_2)_{P_n}\right)\sum_{i(P_3)} B_{P_n i} F(P_3,i)$$

$$+ C(P_1,P_2)_{P_n} \sum_{i(P_n)} B_{P_n i} F(P_n,i) + B^T_{P_1P_n} D(P_1)$$

(4.2-8)

Now we can also get $D(P_n)$ from polyped P_1; substituting equation 4.2-4 we have

$$D(P_n) = - C(P_1,P_n) F(P_3,P_n) + C(P_1,P_n) \sum_{i(P_n)} B_{P_n i} F(P_n,i)$$

$$+ B^T_{P_1P_n} D(P_1) \qquad (4.2\text{-}9)$$

Equating 4.2-8 and 4.2-9, substituting equations I5.2-1D and I5.4-1D and solving for $F(P_3,P_n)$ we have

$$\left(C(P_1,P_2)_{P_n} + C(P_2,P_3)_{P_n} + C(P_3,P_n) + C(P_1,P_n)\right)F(P_3,P_n)$$

$$= C(P_1,P_n)\sum_{i(P_n)} B_{P_n i}\, F(P_n,i) - \left(C(P_1,P_2)_{P_n} + C(P_2,P_3)_{P_n}\right)\cdot$$

$$\cdot\sum_{i(P_3)} B_{P_n i}\, F(P_3,i) - C(P_1,P_2)_{P_n}\sum_{i(P_2)} B_{P_n i}\, F(P_2,i)$$

(4.2-10)

or
$$F(P_3,P_n) = \left((\Sigma C)_{P_n}\right)^{-1} C(P_1,P_n)\sum_{i(P_n)} B_{P_n i}\, F(P_n,i)$$

$$- \left((\Sigma C)_{P_n}\right)^{-1}\left(C(P_1,P_2)_{P_n} + C(P_2,P_3)_{P_n}\right)\sum_{i(P_3)} B_{P_n i}\, F(P_3,i)$$

$$- \left((\Sigma C)_{P_n}\right)^{-1} C(P_1,P_2)_{P_n}\sum_{i(P_2)} B_{P_n i}\, F(P_2,i)$$

(4.2-10.1)

Substituting equation 4.2-10.1 into either equation 4.2-8 or 4.2-9 and simplifying using equations from section I4, then

$$D(P_n) = \left(C(P_1,P_2)_{P_n} + C(P_2,P_3)_{P_n} + C(P_3,P_n)\right)\left((\Sigma C)_{P_n}\right)^{-1}\cdot$$

$$\cdot\, C(P_1,P_n)\sum_{i(P_n)} B_{P_n i}\, F(P_n,i) + C(P_1,P_n)\cdot$$

$$\cdot\left((\Sigma C)_{P_n}\right)^{-1}\left(C(P_1,P_2)_{P_n} + C(P_2,P_3)_{P_n}\right)\sum_{i(P_3)} B_{P\ i}\, F(P_3,i)$$

$$+ C(P_1,P_n)\left((\Sigma C)_{P_n}\right)^{-1} C(P_1,P_2)_{P_n}\sum_{i(P_2)} B_{P_n i}\, F(P_2,i)$$

$$+ C(P_0,P_1)\cdot\sum_{m=P_1}^{P_n}\sum_{i(m)} B_{P_1 i}\, F(m,i)$$

(4.2-11)

Now substituting equation 4.2-10.1 into equation 4.2-3 and inserting indentities 2.1.5-10 and 2.1.5-11 as necessary. Then

$$F(P_2,P_3) = B_{P_3P_n}\left((\Sigma C)_{P_n}\right)^{-1} B^T_{P_3P_n}\ B^T_{P_nP_3}\ C(P_1,P_n)\ B_{P_nP_3}\cdot$$

$$\cdot\sum_{i(P_n)} B_{P_3i}\ F(P_n,i) + \Big[I - B_{P_3P_n}\left((\Sigma C)_{P_n}\right)^{-1} B^T_{P_3P_n}\cdot$$

$$\cdot B^T_{P_nP_3}\left(C(P_1,P_2)_{P_n} + C(P_2,P_3)_{P_n}\right) B_{P_nP_3}\Big]\sum_{i(P_3)} B_{P_3i} F(P_3,i)$$

$$- B_{P_3P_n}\left((\Sigma C)_{P_n}\right)^{-1} B^T_{P_3P_n}\ B^T_{P_nP_3}\ C(P_1,P_2)_{P_n}\ B_{P_nP_3}\cdot$$

$$\cdot\sum_{i(P_2)} B_{P_3i}\ F(P_2,i) \qquad (4.2\text{-}12)$$

Substituting equations I5.2-1D and I5.4-7 we have

$$F(P_2,P_3) = \left((\Sigma C)_{P_3}\right)^{-1} C(P_1,P_n)_{P_3} \sum_{i(P_n)} B_{P_3i}\ F(P_n,i)$$

$$+\Big[I - \left((\Sigma C)_{P_3}\right)^{-1}\left(C(P_1,P_2)_{P_3} + C(P_2,P_3)\right)\Big]\cdot$$

$$\cdot\sum_{i(P_3)} B_{P_3i} F(P_3,i) - \left((\Sigma C)_{P_3}\right)^{-1} C(P_1,P_2)_{P_3} \sum_{i(P_2)} B_{P_3i} F(P_2,i)$$

$$(4.2\text{-}12.1)$$

Now substituting equation 4.2-12.1 into equation 4.2-7.1 and simplifying in the same manner as before, we have

$$D(P_3) = \left(C(P_1,P_2)_{P_3} + C(P_2,P_3)\right)\left((\Sigma C)_{P_3}\right)^{-1} C(P_1,P_n)_{P_3}\cdot$$

$$\cdot\sum_{i(P_n)} B_{P_3i}\ F(P_n,i) + \left(C(P_1,P_2)_{P_3} + C(P_2,P_3)\right)\cdot$$

$$\cdot\left((\Sigma C)_{P_3}\right)^{-1}\left(C(P_1,P_n)_{P_3}+C(P_n,P_3)\right)\sum_{i(P_3)} B_{P_3 i}\ F(P_3,i)$$

$$+\left(C(P_1,P_n)_{P_3}+C(P_n,P_3)\right)\left((\Sigma C)_{P_3}\right)^{-1} C(P_1,P_2)_{P_3}\sum_{i(P_2)} B_{P_3 i} F(P_2,i)$$

$$+\ C(P_0,P_1)\cdot\sum_{m=P_1}^{P_n}\sum_{i(m)} B_{P_1 i}\ F(m,i) \qquad (4.2\text{-}13)$$

Substituting equation 4.2-12.1 into equation 4.2-2 and simplifying in the same manner as before we have

$$F(P_1,P_2) = \left((\Sigma C)_{P_2}\right)^{-1} C(P_1,P_n)_{P_2}\sum_{i(P_n)} B_{P_2 i}\ F(P_n,i)$$

$$+\left[I - \left((\Sigma C)_{P_2}\right)^{-1}\left(C(P_3,P_2)+C(P_1,P_2)\right)\right]\sum_{i(P_3)} B_{P_2 i} F(P_3,i)$$

$$+\left[I - \left((\Sigma C)_{P_2}\right)^{-1} C(P_1,P_2)\right]\sum_{i(P_2)} B_{P_2 i}\ F(P_2,i)$$

$$(4.2\text{-}14)$$

Now substituting equation 4.2-14 into equation 4.2-6 and simplifying in the same manner as before, we have

$$D(P_2) = C(P_1,P_2)\left((\Sigma C)_{P_2}\right)^{-1} C(P_1,P_n)_{P_2}\sum_{i(P_n)} B_{P_2 i}\ F(P_n,i)$$

$$+\ C(P_1,P_2)\left((\Sigma C)_{P_2}\right)^{-1}\left(C(P_1,P_n)_{P_2}+C(P_n,P_3)_{P_2}\right)\cdot$$

$$\cdot\sum_{i(P_3)} B_{P_2 i}\ F(P_3,i) +\ C(P_1,P_2)\left((\Sigma C)_{P_2}\right)^{-1}\cdot$$

$$\cdot\left(C(P_1,P_n)_{P_2}+C(P_n,P_3)_{P_2}+C(P_3,P_2)\right)\sum_{i(P_2)} B_{P_2 i} F(P_2,i)+$$

$$+ C(P_0,P_1)\cdot\sum_{m=P_1}^{P_n}\sum_{i(m)} B_{P_1 i}\ F(m,i) \qquad (4.2\text{-}15)$$

Now using equations 4.2-5, 4.2-15, 4.2-13, and 4.2-11 and using equation 2.6-A we can find the displacements of the external ends of the members of the various polypeds.

Let us simplify these equations using equation 2.6-G, then

$$D_{P_1}(j) = C(P_1,j)\ F(P_1,j) + \sum_{m=P_1}^{P_n}\sum_{i(m)}{}^{*} C_{1_j m_i}(P_0,P_1)\ F(m,i)$$

$$(4.2\text{-}16)$$

$$D_{P_2}(j) = C(P_2,j)\ F(P_2,j) + \sum_{m=P_1}^{P_n}\sum_{i(m)}{}^{*} C_{2_j m_i}(P_0,P_1)\ F(m,i)$$

$$+ B^T_{P_2 j}\Big[C(P_1,P_2)\left((\Sigma C)_{P_2}\right)^{-1}\Big(C(P_1,P_n)_{P_2} +$$

$$C(P_n,P_3)_{P_2} + C(P_3,P_2)\Big)\Big]\sum_{i(P_2)} B_{P_2 i}\ F(P_2,i)$$

$$+ B^T_{P_2 j}\Big[C(P_1,P_2)\left((\Sigma C)_{P_2}\right)^{-1}\Big(C(P_1,P_n)_{P_2} +$$

$$C(P_n,P_3)_{P_2}\Big)\Big]\sum_{i(P_3)} B_{P_2 i}\ F(P_3,i)$$

$$+ B^T_{P_2 j}\Big[C(P_1,P_2)\left((\Sigma C)_{P_2}\right)^{-1}\ C(P_1,P_n)_{P_2}\Big]\sum_{i(P_n)} B_{P_2 i} F(P_n,i)$$

$$(4.2\text{-}17)$$

$$D_{P_3}(j) = C(P_3,j)\ F(P_3,j) + \sum_{m=P_1}^{P_n}\sum_{i(m)}{}^{*} C_{3_j m_i}(P_0,P_1)\ F(m,i)$$

$$+ B^T_{P_3 j}\Big[\Big(C(P_1,P_n)_{P_3} + C(P_n,P_3)\Big)\left((\Sigma C)_{P_3}\right)^{-1}\cdot$$

$$\cdot\, C(P_1,P_2)_{P_3}\Big]\sum_{i(P_2)} B_{P_3 i}\ F(P_2,i) +$$

$$+ B^T_{P_3 j}\left[\left(C(P_1,P_2)_{P_3} + C(P_2,P_3)\right)\left((\Sigma C)_{P_3}\right)^{-1} \cdot\right.$$

$$\left.\cdot\left(C(P_1,P_n)_{P_3} + C(P_n,P_3)_{P_3}\right)\right]\sum_{i(P_3)} B_{P_3 i}\ F(P_3,i)$$

$$+ B^T_{P_3 j}\left[\left(C(P_1,P_2)_{P_3} + C(P_2,P_3)\right)\left((\Sigma C)_{P_3}\right)^{-1} \cdot\right.$$

$$\left.\cdot\ C(P_1,P_n)_{P_3}\right]\sum_{i(P_n)} B_{P_3 i}\ F(P_n,i) \qquad (4.2\text{-}18)$$

and $$D_{P_n}(j) = C(P_n,j)\ F(P_n,j) + \sum_{m=P_1}^{P_n}\sum_{i(m)}{}^{*} C_{n_j m_i}(P_0,P_1)\ F(m,i)$$

$$+ B^T_{P_n j}\left[C(P_1,P_n)\left((\Sigma C)_{P_n}\right)^{-1} C(P_1,P_2)_{P_n}\right]\sum_{i(P_2)} B_{P_n i} F(P_2,i)$$

$$+ B^T_{P_n j}\left[C(P_1,P_n)\left((\Sigma C)_{P_n}\right)^{-1}\left(C(P_1,P_2)_{P_n} + C(P_2,P_3)_{P_n}\right)\right]\cdot$$

$$\cdot \sum_{i(P_3)} B_{P_n i}\ F(P_3,i)$$

$$+ B^T_{P_n j}\left[\left(C(P_1,P_2)_{P_n} + C(P_2,P_3)_{P_n} + C(P_3,P_n)\right)\cdot\right.$$

$$\left.\cdot\left((\Sigma C)_{P_n}\right)^{-1} C(P_1,P_n)\right]\sum_{i(P_n)} B_{P_n i}\ F(P_n,i)$$

$$(4.2\text{-}19)$$

In order to display these equations in a more compact and meaningful form, use equations 3.1-11D and 3.1-13D to define a system displacment and force column matrices. Then we can display the set of equations 4.2-16 through 4.2-19 in the form

$$D = (C_0 + C_1 + G) F \tag{4.2-20}$$

in which C_0 and C_1 are as before, equations 3.1-16.0 and 3.1-16.1 . The G matrix is a flexure matrix in which the partition segments are rather complex combinations of the individual member C matrices.

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & G_{22} & G_{23} & G_{2n} \\ 0 & G_{32} & G_{33} & G_{3n} \\ 0 & G_{n2} & G_{n3} & G_{nn} \end{bmatrix} \tag{4.2-21}$$

where the general term of each G partition segment is

$$G_{2_j 2_i} = B^T_{P_2 j}\, C(P_1,P_2) \left((\Sigma C)_{P_2} \right)^{-1} \left(C(P_1,P_n)_{P_2} + C(P_n,P_3)_{P_2} + C(P_3,P_2) \right) B_{P_2 i} \tag{4.2-21.22}$$

$$G_{2_j 3_i} = B^T_{P_2 j}\, C(P_1,P_2) \left((\Sigma C)_{P_2} \right)^{-1} \left(C(P_1,P_n)_{P_2} + C(P_n,P_3)_{P_2} \right) B_{P_2 i} \tag{4.2-21.23}$$

$$G_{2_j n_i} = B^T_{P_2 j}\, C(P_1,P_2) \left((\Sigma C)_{P_2} \right)^{-1} C(P_1,P_n)_{P_2}\, B_{P_2 i} \tag{4.2-21.2n}$$

$$G_{3_j 2_i} = B^T_{P_3 j} \left(C(P_1,P_n)_{P_3} + C(P_n,P_3) \right) \left((\Sigma C)_{P_3} \right)^{-1} C(P_1,P_2)_{P_3}\, B_{P_3 i} \tag{4.2-21.32}$$

$$G_{3_j 3_i} = B^T_{P_3 j}\left(C(P_1,P_2)_{P_3} + C(P_2,P_3)\right)\left((\Sigma C)_{P_3}\right)^{-1} \cdot$$

$$\cdot\left(C(P_1,P_n)_{P_3} + C(P_n,P_3)\right)B_{P_3 i} \quad (4.2\text{-}21.33)$$

$$G_{3_j n_i} = B^T_{P_3 j}\left(C(P_1,P_2)_{P_3} + C(P_2,P_3)\right)\left((\Sigma C)_{P_3}\right)^{-1} \cdot$$

$$\cdot C(P_1,P_n)_{P_3} \; B_{P_3 i} \quad (4.2\text{-}21.3n)$$

$$G_{n_j 2_i} = B^T_{P_n j} \; C(P_1,P_n)\left((\Sigma C)_{P_n}\right)^{-1} C(P_1,P_2)_{P_n} \; B_{P_n i}$$

$$(4.2\text{-}21.n2)$$

$$G_{n_j 3_i} = B^T_{P_n j} \; C(P_1,P_n)\left((\Sigma C)_{P_n}\right)^{-1}\left(C(P_1,P_2)_{P_n} +\right.$$

$$\left. C(P_2,P_3)_{P_n}\right)B_{P_n i} \quad (4.2\text{-}21.n3)$$

$$G_{n_j n_i} = B^T_{P_n j}\left(C(P_1,P_2)_{P_n} + C(P_2,P_3)_{P_n} + C(P_3,P_n)\right)\left((\Sigma C)_{P_n}\right)^{-1} \cdot$$

$$\cdot C(P_1,P_n) \; B_{P_n i} \quad (4.2\text{-}21.nn)$$

Thus we have described the displacment of all the external members of all the polypeds in terms of forces applied to the various members of the polypeds. This set of linear equations, although certain members appear to be quite complicated, may be solved in a straightforward manner. Let us leave the actual solution to the algebraist. However, before we leave this problem we should investigate the G matrix further.

4.3. The Equivalent Flexure Matrix of a Loop of Polypeds.

The set of equations 4.2-21.RS represent the general term of each submatrix or partition segment of the G matrix, the equivalent flexure matrix of the loop of polypeds. Now, in order for equation 4.2-20 to obey the reciprocal theorems of structural theory, the matrix $(C_0 + C_1 + G)$ must be symmetric. We have already seem that C_0 and C_1 are symmetric. Therefore G must be symmetric. Let us show this symmetry.

Let us consider some arbitrary point o. Write the G matrix in the form

$$G = B_o^T G' B_o \qquad (4.3\text{-}1)$$

in which $B_o = \text{diag}\left[B_{o1}, B_{o2}, B_{o3}, B_{on}\right] \qquad (4.3\text{-}1.1)$

each of the terms in equation 4.3-1.1 is itself a diagonal matrix of n_{P_i} terms. The individual terms represent the transformation from the particular external end of the member to the arbitrary point o.

Let us consider in detail what form the G' matrix will have. For example consider in particular the partition segment G_{n3}. If equation 4.2-21.n3 is rewritten using the identity relations 2.1.5-10 and 2.1.5-11

$$G_{n_j 3_i} = B_{on_j}^T B_{P_n o}^T C(P_1, P_n) B_{P_n o} B_{oP_n} \left((\Sigma C)_{P_n}\right)^{-1} B_{oP_n}^T$$

$$B_{P_n o}^T \left(C(P_1, P_2)_{P_n} + C(P_2, P_3)_{P_n} \right) B_{P_n o} B_{o3_i} \qquad (4.3\text{-}2)$$

As an extension of equation I5.4-7, which is applicable only for translations along members to other nuclei of the loop of polypeds, let us define the following

$$(\Sigma C)_{jo} = B^T_{P_j o} \, (\Sigma C)_{P_j} \, B_{P_j o} \qquad (4.3\text{-}3D)$$

$$= C(P_1,P_2)_o + C(P_2,P_3)_o + \ldots + C(P_{j-1},P_j)$$

$$+ C(P_1,P_n)_o + C(P_n,P_3)_o + \ldots + C(P_{j\ 1},P_j)$$

$$(4.3\text{-}3.1)$$

In the term $(\Sigma C)_{jo}$ the first subscript indicates the point to which the various C matrices were translated, the second subscript indicates that subsequently the various C matrices were translated to an arbitrary point o. As indicated in the note following equation I5.4-6 the first subscript is needed to indicate the order of subscripts in the individual C matrices of the sum.

Regrouping the terms using the equations 4.3-3D and I5.2-1D, then we have

$$G_{n\ 3} = B^T_{on_j} \, C(P_1,P_n)_o \left((\Sigma C)_o\right)^{-1} C(P_1,P_2)_o \quad C(P_2,P_3)_o \; B_{o3_i}$$

$$(4.3\text{-}4)$$

This same procedure is applied to each term of the G matrix. We can indeed write the matrix G in the form of equation 4.3-1, in which G' has the form

$$G' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & G'_{22} & G'_{23} & G'_{2n} \\ 0 & G'_{32} & G'_{33} & G'_{3n} \\ 0 & G'_{n2} & G'_{n3} & G'_{nn} \end{bmatrix} \tag{4.3-5}$$

in which the general term of each partition segment is

$$G'_{22} = C(P_1,P_2)_o \left((\Sigma C)_o\right)^{-1} \left(C(P_1,P_n)_o + C(P_n,P_3)_o + C(P_3,P_2)_o\right) \tag{4.3-5.22}$$

$$G'_{23} = C(P_1,P_2)_o \left((\Sigma C)_o\right)^{-1} \left(C(P_1,P_n)_o + C(P_n,P_3)_o\right) \tag{4.3-5.23}$$

$$G'_{2n} = C(P_1,P_2)_o \left((\Sigma C)_o\right)^{-1} C(P_1,P_n)_o \tag{4.3-5.2n}$$

$$G'_{32} = \left(C(P_1,P_n)_o + C(P_n,P_3)_o\right)\left((\Sigma C)_o\right)^{-1} C(P_1,P_2)_o \tag{4.3-5.32}$$

$$G'_{33} = \left(C(P_1,P_2)_o + C(P_2,P_3)_o\right)\left((\Sigma C)_o\right)^{-1} \left(C(P_1,P_n)_o + C(P_n,P_3)_o\right) \tag{4.3-5.33}$$

$$G'_{3n} = \left(C(P_1,P_2)_o + C(P_2,P_3)_o\right)\left((\Sigma C)_o\right)^{-1} C(P_1,P_n)_o \tag{4.3-5.3n}$$

$$G'_{n2} = C(P_1,P_n)_o \left((\Sigma C)_o\right)^{-1} C(P_1,P_2)_o \tag{4.3-5.n2}$$

$$G'_{n3} = C(P_1,P_n)_o \left((\Sigma C)_o\right)^{-1} \left(C(P_1,P_2)_o + C(P_2,P_3)_o\right) \tag{4.3-5.n3}$$

$$G'_{nn} = \left(C(P_1,P_2)_o + C(P_2,P_3)_o + C(P_3,P_n)_o\right)\left(\Sigma C)_o\right)^{-1} C(P_1,P_n)_o$$

(4.3-5.nn)

Let us consider the symmetry of the G' matrix. We will consider the off-diagonal terms first. Using equation I3.4-1, which states that the transpose of a product reverses the order of the terms, and equation I5.3-3, which states that the transpose of a translated flexure matrix is the flexure matrix itself; it is obvious that

$$G'_{RS} = G'_{SR}{}^{T} \qquad (\text{for } R \neq S)$$

Now we will consider the diagonal terms. Note that all the diagonal terms are of the form $A(A+B)^{-1}B$. If we consider the transpose of G'_{RR}, the result is of the form $B(A+B)^{-1}A$. Using the equations I4.1-1 and I4.1-2 we see that the above forms are equivalent, and thus the diagonal terms are also their own transpose. Thus we find that the G' matrix is symmetric.

If G' is symmetrical, then using the matrix property, in section I 3.8, the matrix G is also symmetrical.

In computational use the computer technique used will dictate whether the use of the G or the G' matrix is more useful.

Thus far we have not considered if the G' or G matrix has any pattern in the C matrices from which it is composed. Let us investigate the G' matrix for such patterns. Inspection of the various equations 4.3-5.RS will lead us to the following empirical statements:

1. The first portion of each term is the sum of all the flexure matrices which form a linkage from the polyped where the displacement is observed to the polyped which links to the fixed point.

2. The middle portion of each term is a summation of the flexure matrices of all the links which make up the loop of polypeds.

3. The last portion of the term is a sum of all the flexure matrices which form a linkage from the polyped where the force is applied to the polyped which links to the fixed point.

4. All the flexure matrices are translated to the arbitrary point o.

5. As we saw earlier in this section the diagonal terms may be written in either of two forms, which are equivalent. That is, from the polyped where the forces act and the displacements are observed there are two distinct paths to the polyped which links to the fixed point. We may write the sum of all the flexure matrices which make up one path either as the first or last portion of the G' term, and then the sum of the flexure matrices in the other path goes in the opposite position.

We are now in a position to generalize the G' matrix to any number of polypeds in the loop. A tedious analysis similiar to the one in section 4.2 will verify that the above statements are true for an n-loop of polypeds.

We have thus described the displacement of any external

end of any polyped in terms of forces applied to any external end of a loop of polypeds. We have derived the rules for formulating the equivalent flexure matrix of the loop of polypeds. We can now use these results to replace a loop of polypeds with a complex polyped having the appropriate equivalent flexure matrix.

CHAPTER 5

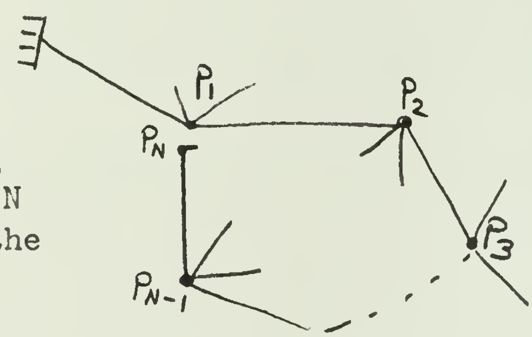
Another Approach to Loop Connected Polypeds

5. It is possible to consider loop connected polypeds, not as two parallel chains of polypeds, but as one long chain of polypeds with the last polyped of the chain connected onto an earlier polyped of the chain. In the method of solution suggested here we assume that the solution for the chain of polypeds is known, i.e., we have a solution of the type derived in chapter 3 for the long chain of polypeds.

5.1. <u>Chain of Polypeds with Last Member Connected to an Earlier Polyped.</u>

Without loss of generality we can have the last member connected back to the first polyped of the chain of polypeds under consideration. If this is not the case then we consider here only the last N polypeds of the chain and will consider in the next chapter how to connect this loop of polypeds to the chain of polypeds which we neglect here. Consider the following sketch.

The points P_1 and P_N are geometrically the same point.



We have the solution for the chain of polypeds, from

equation 3.1-15 which we may express in the form

$$D' = (C_0 + C_1 + \ldots + C_N) F' \qquad (5.1\text{-}1)$$

where $D' = \left\{ D_{P_1}, D_{P_2}, \ldots, D_{P_N} \right\}$ from (3.1-11D)

and $F' = \left\{ F(P_1,i), \ldots, F(P_N,i) \right\}$ from (3.1-13D)

and each of the D_{P_j} and $F(P_j,i)$ submatrices is in turn a column matrix of the displacements (or forces) on the n_{P_j} external ends of the polyped P_j. The C_k are given by the equations 3.1-16.k.

To the polyped P_1 let us add another member, say r, wich is rigid and is of infinitesimal length. Then the displacement and force column matrices for the polyped P_1 will be altered as follows:

$$D_{P_1} \text{ becomes } \left\{ D(r), D(1), D(2), \ldots, D(n_{P_1}) \right\}$$
$$= \left\{ D(r), D_{P_1} \right\} \qquad (5.1\text{-}2)$$

$$\text{and } F(P_1,i) \text{ becomes } \left\{ F(P_1,r), F(P_1,1), \ldots, F(P_1,n_{P_1}) \right\}$$
$$= \left\{ F(P_1,r), F(P_1,i) \right\} \qquad (5.1\text{-}3)$$

Accordingly the column matrices D' and F' in equation 5.1-1 become

$$D'' = \left\{ D(r), D_{P_1}, D_{P_2}, \ldots, D_{P_N} \right\} \qquad (5.1\text{-}4)$$

$$F'' = \left\{ F(P_1,r), F(P_1,i), \ldots, F(P_N,i) \right\} \qquad (5.1\text{-}5)$$

Also the various C_k of equation 5.1-1 must be augmented. We augment each of the matrices with a row and column of

6 x 6 matrices at the top and left side, coresponding to the r-th partition we have made in the column matrices D" and F". In order not to disturb the numbering scheme for the submatrices previously used let us denote this additional row and column by the subscript $_0$. Using the argument of section 2.5 we note that in C_0 this row and column will be composed of zero matrices. In C_1 the terms will have the value given by the equation 2.5-2; thus in the newly added portion of C_1 we have

$$C_{j0} = B^T_{P_1 j} \; C(P_0,P_1) \qquad (5.1\text{-}6.1)$$

$$C_{0i} = C(P_0,P_1) \; B_{P_1 i} \qquad (5.1\text{-}6.2)$$

In C_2 and subsequent C_k the added row and column will be composed of zero matrices since the submatrices corresponding to members of the first polyped are already zero.

Let us now express the matrix equation 5.1-1 in the form

$$D'' = S \, F'' \qquad (5.1\text{-}7)$$

where

$$S = \begin{bmatrix} S_{00} & S_{01} & S_{02} & \cdots & S_{0N} \\ S_{10} & S_{11} & S_{12} & \cdots & S_{1N} \\ \cdots & & & & \\ S_{N0} & S_{N1} & S_{N2} & \cdots & S_{NN} \end{bmatrix} \qquad (5.1\text{-}7.1)$$

and each of the S_{RS} submatrices is the sum of the C_{RS} submatrices of each of the C_k matrices. If we had forseen

the need for a rigid infinitesimal member r on polyped P_1 when we solved the chain of polypeds, we could have made the partitioning of the column matrix for the force and displacement in polyped P_1 as shown in equations 5.1-2 and 5.1-3 . We then would have equation 5.1-7 without further ado.

Let us write the individual equations represented by the matrix equation 5.1-7 . (For the following algebra let us shorten $F(P_j,i)$ to F_{P_j}, etc.)

$$D(r) = S_{00}F_r + S_{01}F_{P_1} + S_{02}F_{P_2} + \dots + S_{0N}F_{P_N} \qquad (5.1\text{-}8.0)$$

$$D_{P_1} = S_{10}F_r + S_{11}F_{P_1} + S_{12}F_{P_2} + \dots + S_{1N}F_{P_N} \qquad (5.1\text{-}8.1)$$

$$\dots$$

$$D_{P_j} = S_{j0}F_r + S_{j1}F_{P_1} + S_{j2}F_{P_2} + \dots + S_{jN}F_{P_N} \qquad (5.1\text{-}8.j)$$

$$\dots$$

$$D_{P_N} = S_{N0}F_r + S_{N1}F_{P_1} + S_{N2}F_{P_2} + \dots + S_{NN}F_{P_N} \qquad (5.1\text{-}8.N)$$

Notice from the sketch that the only external member of polyped P_N is a rigid, infinitesimal member. In order to make a loop let

$$D_{P_1}(r) = D_{P_N}(1) \qquad (5.1\text{-}9)$$

or since the above quantities represent the only members in perticular partition terms of equation 5.1-4 we have

$$D(r) = D_{P_N} \qquad (5.1\text{-}9.1)$$

Also in making this loop junction we must satisfy the equilibrium of forces and thus

$$F(P_N,i) + F(P_1,r) = 0 \qquad (5.1\text{-}10)$$

or $$F_r = - F_{P_N} \qquad (5.1\text{-}10.1)$$

Substituting equation 5.1-10.1 into the various equations 5.1-8.j we have

$$D(r) = S_{01}F_{P_1} + S_{02}F_{P_2} + \cdots + (S_{0N} - S_{00})\, F_{P_N} \qquad (5.1\text{-}11.0)$$

$$D_{P_1} = S_{11}F_{P_1} + S_{12}F_{P_2} + \cdots + (S_{1N} - S_{10})\, F_{P_N} \qquad (5.1\text{-}11.1)$$

$$\cdots$$

$$D_{P_j} = S_{j1}F_{P_1} + S_{j2}F_{P_2} + \cdots + (S_{jN} - S_{j0})\, F_{P_N} \qquad (5.1\text{-}11.j)$$

$$\cdots$$

$$D_{P_N} = S_{N1}F_{P_1} + S_{N2}F_{P_2} + \cdots + (S_{NN} - S_{N0})\, F_{P_N} \qquad (5.1\text{-}11.N)$$

Now if we substract equation 5.1-11.N from 5.1-11.0 we have

$$0 = (S_{01} - S_{N1})\, F_{P_1} + (S_{02} - S_{N2})\, F_{P_2} + \cdots$$

$$+ \Big((S_{0N} - S_{00}) - (S_{NN} - S_{N0})\Big)\, F_{P_N} \qquad (5.1\text{-}12)$$

To simplify the algebra let us define a matrix H

$$H = S_{00} - S_{0N} - S_{N0} + S_{NN} \qquad (5.1\text{-}13D)$$

Premultiplying equation 5.1-12 by H^{-1} we have

$$0 = H^{-1}(S_{01} - S_{N1})F_{P_1} + H^{-1}(S_{02} - S_{N2})F_{P_2} + \cdots - F_{P_N}$$

(5.1-14)

Premultiplying equation 5.1-14 by $(S_{jN} - S_{j1})$ and adding the result sucessively to each of the equations 5.1-11.1 through N-1 we will eliminate F_{P_N} from each equation. Thus

$$D_{P_1} = \Big(S_{11} + (S_{1N} - S_{11})\, H^{-1}\, (S_{01} - S_{N1})\Big)F_{P_1} +$$

$$\Big(S_{12} + (S_{1N} - S_{11})\, H^{-1}\, (S_{02} - S_{N2})\Big)F_{P_2} + \cdots$$

$$+\Big(S_{1i} + (S_{1N} - S_{11})\, H^{-1}\, (S_{0i} - S_{Ni})\Big)F_{P_i} + \cdots + (0)F_{P_N}$$

(5.1-15.1)

...

$$D_{P_j} = \Big(S_{j1} + (S_{jN} - S_{j1})\, H^{-1}\, (S_{01} - S_{N1})\Big)F_{P_1} +$$

$$\Big(S_{j2} + (S_{jN} - S_{j1})\, H^{-1}\, (S_{02} - S_{N2})\Big)F_{P_2} + \cdots$$

$$\Big(S_{ji} + (S_{jN} - S_{j1})\, H^{-1}\, (S_{0i} - S_{Ni})\Big)F_{P_i} + \cdots + (0)F_{P_N}$$

(5.1-15.j)

...

The set of equations 5.1-15.j may be compacted in the form

$$D = S^{*} F \qquad (5.1\text{-}16)$$

where $D = \left\{D_{P_1},\ D_{P_2},\ \ldots\ ,\ D_{P_{N-1}}\right\}$ (5.1-16.1)

and $F = \left\{ F(P_1,i),\ F(P_2,i),\ \dots\ ,\ F(P_{N-1},i) \right\}$

$$(5.1\text{-}16.2)$$

and $S^{*}_{ji} = S_{ji} + (S_{jN} - S_{j1})\ H^{-1}\ (S_{0i} - S_{Ni})$ $\quad(5.1\text{-}17)$

Note that S_{0i} is given by equation 5.1-6.2. Using the equations 5.1-6.1 and 5.1-6.2, the expression for H becomes

$$H = S_{NN} - C(P_0,P_1) \qquad (5.1\text{-}18)$$

The additional member r which we added to the polyped P_1 has been eliminated as a pertinent point. The solution to the loop of polypeds is thus expressed in terms of the submatrices of a chain of polypeds but in a more complicated form.

CHAPTER 6

Generalizations

6. In the preceding chapters we have developed methods to find the flexure matrix for a complicated system of members. We have considered in detail chains of polypeds and loops of polypeds. Very complicated structures can be disjoined into these two types of polyped groupings. We have what we shall call an "equivalent polyped" to replace a chain or loop of polypeds. The equivalent polypeds then become the members of still larger polypedlike structures. That is to say we develop larger and larger entities, starting with members; we combine members into simple polypeds, then we combine these simple polypeds into equivalent polypeds, and these equivalent polypeds may then be combined into still larger equivalent polypeds.

We have been interested in the B-transformation which translate the flexure (equivalent flexure) matrix from the end of the (equivalent) member to the pertinent point, either to determine the displacement there, or to apply a force at that point. These B-transformations are geometric transformations and are not concerned with whether the translation is along a member or just a translation through space.

In the manner in which we use the flexure matrices it is obvious upon reflection that the flexure matrix could equally well have been either a simple flexure matrix or an equivalent matrix and the applicability of the formulas derived is unaltered.

6.1. Generalizing the "Fixed Point".

In the preceding chapters we have considered some point in each system under discussion to be fixed. If however, we allowed this point to move, then we must add additional terms to the overall solution. Consider for example, a chain of polypeds with the so called fixed point having a displacement $D(P_0)$. Then following the pattern of solution in section 3.1 , equation 3.1-4 would be

$$D(P_1) = C(P_0,P_1)\ F(P_0,P_1) + B^T_{P_0 P_1}\ D(P_0) \tag{6.1-1}$$

and then equation 3.1-8.1 would become

$$D_{P_1}(j) = C(P_1,j)\ F(P_1,j) + \sum_{m=P_1}^{P_n} \sum_{i(m)} {}^{*}C_{1_j m_i}(P_0,P_1)\ F(m,i) + B^T_{P_0 j}\ D(P_0) \tag{6.1-2}$$

and similiarly, the equations for the other external displacements will merely add the term

$$+ B^T_{P_0 j}\ D(P_0)\ .$$

Thus we can see that permitting the previously fixed point to move merely adds a term representing the motion of the point P_0 .

If we consider a loop of polypeds and allow the "fixed" point to have the displacement $D(P_0)$, the same results are obtained, namely that all the external member end displace-displacements in each of the various polypeds have an addit-

ional term due to the rigid body displacement of P_0 translated to the pertinent point.

6.2. Connecting Two Chains of Polypeds.

Let us now consider a chain of polypeds N-members long. Let us assume it is fixed; however, if it is not we can treat it as in section 6.1 . To the end of this N-member chain let us attach another chain of polypeds, M-members long. Let us denote the members of the M-member chain by ' .

Then we have for the displacement of the nucleus of the first member of the second chain

$$D(P_{1'}) = C(P_N,P_{1'})\ F(P_N,P_{1'}) + B^T_{P_N P_{1'}}\ D(P_N) \tag{6.2-1}$$

similiarly in the equation for the displacement of each nucleus and pertinent point we have added a term of the form

$$+ B^T_{P_N P_{j'}}\ D(P_N)$$

If we expand the $D(P_N)$ term in each of the equations for the displacement of the pertinent points of the M polypeds we will find that each equation will have a group of terms characterizing the effect of the forces in the first N polypeds. For example let us exhibit $D_{P_{k'}}(j)$

$$D_{P_{k'}}(j) = C(P_{k'},j)\ F(P_{k'},j) + \sum_{m=P_1}^{P_N} \sum_{i(m)}{}^{*} C_{k'_j m_i}(P_0,P_1)\ F(m,i)$$
$$+ \sum_{m=P_1}^{P_M} \sum_{i(m)}{}^{*} C_{k'_j m_i}(P_0,P_1)\ F(m,i) + \cdots\cdot$$

$$+\sum_{m=P_N}\sum_{i(m)}{}^{*}C_{k'_j m_i}(P_{N-1},P_N)F(m,i)+\sum_{m=P_1'}^{P_M}\sum_{i(m)}{}^{*}C_{k'_j m_i}(P_{N-1},P_N)F(m,i)$$

$$+\sum_{m=P_1'}^{P_M}\sum_{i(m)}{}^{*}C_{k'_j m_i}(P_N,P_{1'})F(m,i)+\cdots+\sum_{m=P_M}\sum_{i(m)}{}^{*}C_{k'_j m_i}(P_{M-1},P_M)F(m,i)$$

(6.2-2.k')

We have here the appropriate terms for the k'-th row of each of the C_k matrices.

Also we find that the forces in the members of the M polypeds affect the displacements of the various members of the N polypeds. We have

$$F(P_{N-1},P_N) = \sum_{i(P_N)} B_{P_N i}\, F(P_N,i) + B_{P_N P_{1'}} F(P_N,P_{1'})$$

(6.2-3)

which in terms only of the pertinent points becomes

$$F(P_{N-1},P_N) = \sum_{i(P_N)} B_{P_N i}\, F(P_N,i) + \sum_{m=P_1'}^{P_M}\sum_{i(m)} B_{P_N m}\, F(m,i)$$

(6.2-3.1)

If we follow through the algebra of section 3.1 again, substituting equation 6.2-3.1 as appropriate, then we have for the result a set of equations similiar to equation 3.1-10.k , but in a generalized form. Let us exhibit the set of equations for the displacement of pertinent points of the k-th polyped of the first chain.

$$D_{P_k}(j) = C(P_k, j)\, F(P_k, j) +$$

$$+ \sum_{m=P_1}^{P_N} \sum_{i(m)}{}^{*} C_{k_j m_i}(P_0, P_1) F(m,i) + \sum_{i=P_1'}^{P_M} \sum_{i(m)}{}^{*} C_{k_j m_i}(P_0, P_1) F(m,i)$$

$$+ \sum_{i=P_2}^{P_N} \sum_{i(m)}{}^{*} C_{k_j m_i}(P_1, P_2) F(m,i) + \sum_{i=P_1'}^{P_M} \sum_{i(m)}{}^{*} C_{k_j m_i}(P_1, P_2) F(m,i)$$

$$+ \quad \cdots + \quad \sum_{m=P_N} \sum_{i(m)}{}^{*} C_{k_j m_i}(P_{N-1}, P_N)\, F(m,i)$$

$$\sum_{m=P_1'}^{P_M} \sum_{i(m)}{}^{*} C_{k_j m_i}(P_{N-1}, P_N)\, F(m,i) \qquad (6.2\text{-}4.k)$$

Notice that here also we have the appropriate terms for the appropriate rows of the various C_k matrices.

When we consider the sets of equations 6.2-2.k' and 6.2-4.k we find that we can compact the equations into the form

$$D = (C_0 + C_1 + C_2 + \ldots + C_N + C_{N+1} + \ldots + C_{N+M})\, F \qquad (6.2\text{-}5)$$

in which D and F are column matrices having first the N polyped pertinent points and then the M polyped pertinent points as matrix elements. Each of the C_k is an $(N+M) \times (N+M)$ partitioned matrix. The additional rows and columns on the C_0 through C_N matrices represent the effect of attaching the M-member chain onto a movable point at the end of the N-member polyped chain instead of the fixed point.

As before, each of the C_k , $(k>1)$, has one more row and column of zero matrices than the one before it. The C_k is bordered on the left and top by k-1 columns and rows of zero matrices. This should be obvious since only the forces in the polypeds farther from the real fixed point than the linkage from polyped k-1 to polyped k affect the displacement of pertinent points in polypeds farther from the real fixed point than polyped k-1 by causing distortions of the linkage from k-1 to k. The remaining submatrix terms are the appropriate $^*C$ type terms to indicate the distortion in the linkage from the k-1 to the k-th polyped caused by these forces farther away from the fixed point than the k-1 polyped.

The above equation, 6.2-5, is exactly the results which would have been obtained if we had considered the structure as a chain $N+M$ members long. Thus we can consider a long chain of polypeds as the result of several shorter ones if that method of analysis happens to be easier to consider and calculate.

6.3. Adding a Loop of Polypeds to a Chain of Polypeds.

Let us consider attaching an M-member loop of polypeds onto the end of an N-member chain of polypeds. As before we may assume that the N-member chain is fixed; if it is not, see section 6.1. Let us denote the members of the M-member loop by ' .

Equation 4.2-5, which gives the displacement of the nucleus of the first member of the loop of polypeds (assuming

that P_0 is the same point as P_N) becomes

$$D(P_{1'}) = C(P_N,P_{1'}) \cdot \sum_{m=P_1'}^{P_M} \sum_{i(m)} B_{P_{1'},i} \; F(m,i) + B^T_{P_N P_{1'}} \; D(P_N)$$

(6.3-1)

Similiarly we would have an added term in the displacement of each nucleus in the loop of polypeds of the form

$$+ B^T_{P_N P_{j'}} \; D(P_N)$$

and a similiar term in the displacement of the external end-points. If we expand the $D(P_N)$ term in each of the equations in the M-members of the loop of polypeds, as before, we will find that we have a group of terms characterizing the effect of the force in the N-polypeds of the chain of polypeds. For example, in the k'-th member of the loop we have (the terms of the G matrix will not be exhibited; however, they are explicitely stated in equation 4.2-21)

$$D_{P_{k'}}(j) = C(P_{k'},j) \; F(P_{k'},j) + \sum_{m=P_1}^{P_N} \sum_{i(m)}{}^{*} C_{k_j m_i}(P_0,P_1) \; F(m,i)$$

$$+ \sum_{m=P_1'}^{P_M} \sum_{i(m)}{}^{*} C_{k_j m_i}(P_0,P_1) \; F(m,i) + \cdots\cdots$$

$$+ \sum_{m=P_N} \sum_{i(m)}{}^{*} C_{k_j m_i}(P_{N-1},P_N) F(m,i) + \sum_{m=P_1'}^{P_M} \sum_{i(m)}{}^{*} C_{k_j m_i}(P_{N-1},P_N) F(m,i)$$

$$+ \sum_{m=P_1'}^{P_M} \sum_{i(m)}{}^{*} C_{k_j m_i}(P_N,P_{1'}) F(m,i) + \sum_{m=P_2'}^{P_M} \sum_{i(m)} G_{k_j m_i} \; F(m,i)$$

(6.3-2.k')

We have here the appropriate terms for the k'-th row of the various C_k matrices and the G matrix.

Also we find that the forces in the M-polypeds affect the displacement of the members of the N-polypeds. Thus

$$F(P_{N-1},P_N) = \sum_{i(P_N)} B_{P_N i} F(P_N,i) + B_{P_N P_{1'}} F(P_N,P_{1'})$$

(6.3-3)

Since P_N is the same point as P_0, we can substitute equations 4.2-1 through 4.2-4. Expressed only in terms of pertinent points we have

$$F(P_{N-1},P_N) = \sum_{i(P_N)} B_{P_N i} \ F(P_N,i) + \sum_{m=P_1'}^{P_M} \sum_{i(m)} B_{P_N m} \ F(m,i)$$

(6.3-4)

As in section 6.2, if we follow through the algebra of section 3.1 again we will arrive at a set of equations for the displacement of the external ends of the various members of the N-polypeds. These equations are the same as equations 6.2-4.k .

When we consider the set of equations 6.2-4.k and 6.3-2.k' we find that we can compact the equations into the form

$$D = (C_0 + C_1 + C_2 + \ldots + C_N + C_{N+1} + G)\ F$$

(6.3-5)

in which D and F are column matrices having first the N polypeds' pertinent points and then the M polypeds' pertinent points as matrix elements. Each of the C_k is an

$(N+M) \times (N+M)$ partitioned matrix. The C_0 through C_N matrices are the same as in the previous section. The C_{N+1} matrix represents the effect of the various forces in the loop of polypeds on the elastic distortion of the member from the nucleus of the N-th polyped of the chain to the nucleus of the first member in the loop of polypeds. This displacement is observed only in the M polypeds of the loop of polypeds. Thus the C_{N+1} matrix is bordered at the top and on the left by N rows and columns of zero matrix segments. This is the C_1 matrix of a loop of polypeds solution augmented by N rows and columns of zero matrix segments. The G matrix for a loop of polypeds by itself is now bordered by N more rows and columns of zero matrix segments. Thus we can see that by knowing the solution of a chain of polypeds and the solution of a loop of polypeds we can immediately write the appropriate terms of a combined solution:

1. The C_0 matrices for the chain and loop solution are combined into a larger diagonal matrix.

2. The various C_k matrices for the chain of polypeds are augmented below and to the right with M rows and columns of matrix elements. The top and left k-1 terms in a row or column are zero matrices, the other elements have the same form as the general element of the C_k matrix for a chain of polypeds.

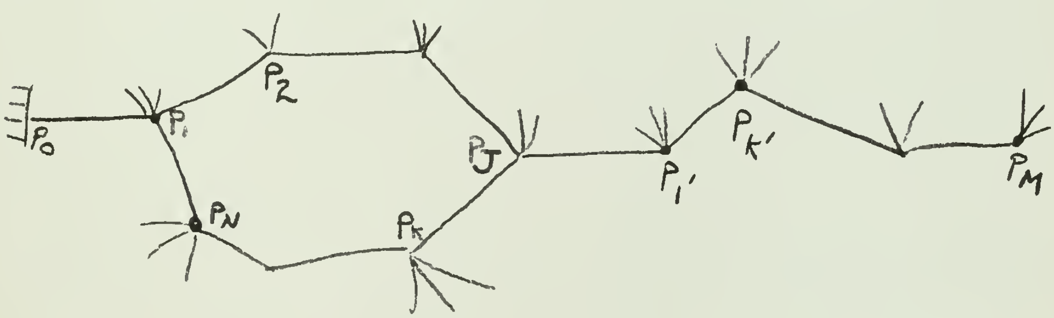
3. The C_{N+1} matrix is the C_1 matrix for the loop of polypeds bordered at the top and left by N rows and columns

of zero matrix elements.

4. The G matrix for the loop of polypeds is bordered at the top and left be N rows and columns of zero matrix elements.

6.4. Adding a Chain of Polypeds to a Loop of Polypeds.

Let us consider adding an M-member chain of polypeds to some nucleus, say P_J, of a loop of N-polypeds. As before we may assume the N member loop of polypeds is fixed, otherwise we have additional terms as per section 6.1. Let us denote the M polyped chain members by ' . Consider the following sketch for clarity.



For the displacement of the nucleus of the J-th polyped in the loop of polypeds, when considered alone, we have

$$D(P_J) = \sum_{n=P_1}^{P_N} \sum_{i(n)}^{*} C_{Jn_i}(P_0,P_1)F(n,i) + \sum_{n=P_1}^{P_N} \sum_{i(n)} B^T_{oJ} G'_{Jn_i} B_{on_i} F(n,i) \tag{6.4-1}$$

When we consider attaching the chain of polypeds at the nucleus P_J we must consider a new force equilibrium. Thus at the nucleus P_J we have for the force equilibrium

$$F(P_{J-1},P_J) + F(P_{J+1},P_J) = \sum_{i(P_J)} B_{P_J i} F(P_J,i) + \sum_{m=P_1'}^{P_M} \sum_{i(m)} B_{P_J i} F(m,i)$$

(6.4-2)

Using this force equilibrium, following through the algebra of section 4.2 considering the effect of the forces in the chain of polypeds, we find for the displacement of the $D(P_J)$

$$D(P_J) = \sum_{n=P_1}^{P_N} \sum_{i(n)}{}^{*} C_{Jn_i}(P_0,P_1) F(n,i) + \sum_{n=P_1}^{P_N} \sum_{i(n)} B^T_{oJ}\, G'_{Jn_i}\, B_{on_i} F(n,i)$$

$$+ \sum_{m=P_1'}^{P_M} \sum_{i(m)} B^T_{oJ}\, G'_{JJ}\, B_{om_i}\, F(m,i) \qquad (6.4\text{-}3)$$

For the displacement of a pertinent point of one of the M polypeds, if we consider $D(P_J)$ as the "fixed" point of a chain of polypeds, we have for the general member of the k' polyped

$$D_{P_{k'}}(j) = C(P_{k'},j)\, F(P_{k'},j) + \sum_{m=P_1'}^{P_M} \sum_{i(m)}{}^{*} C_{k'_j m_i}(P_J,P_{1'})\, F(m,i)$$

$$+ \sum_{m=P_2'}^{P_M} \sum_{i(m)}{}^{*} C_{k'_j m_i}(P_{1'},P_{2'})\, F(m,i) + \cdots\cdot$$

$$+ \sum_{m=P_M} \sum_{i(m)}{}^{*} C_{k'_j m_i}(P_{M-1},P_M)\, F(m,i) + B^T_{P_J P_{k'}}\, D(P_J)$$

(6.4-4)

Combining equations 6.4-3 and 6.4-4 we have

$$D_{P_{k'}}(j) = C(P_{k'},j)\, F(P_{k'},j) + \sum_{m=P_1'}^{P_M} \sum_{i(m)}{}^{*} C_{k'_j m_i}(P_J,P_{1'})\, F(m,i) +$$

$$+\sum_{m=P_{2'}}^{P_M}\sum_{i(m)}{}^{*}C_{k'_j m_i}(P_{1'},P_{2'})\ F(m,i)+\ \ldots.$$

$$+\sum_{m=P_M}\sum_{i(m)}{}^{*}C_{k'_j m_i}(P_{M-1},P_M)F(m,i)+\sum_{n=P_1}^{P_N}\sum_{i(n)}{}^{*}C_{k'_j n_i}(P_0,P_1)\ F(n,i)$$

$$+\sum_{n=P_1}^{P_N}\sum_{i(n)}B^T_{ok'_j}\ G'_{Jn_i}\ B_{on_i}\ F(n,i)+\sum_{m=P_1}^{P_M}\sum_{i(m)}B^T_{ok'_j}\ G'_{JJ}\ B_{om_i}\ F(m,i)$$

(6.4-5.k')

Now if we use equation 6.4-2 and follow through the tedious algebra of section 4.2, then for the displacement of the j-th external end of a member of the k-th polyped in the loop of polypeds we have

$$D_{P_k}(j) = C(P_k,j)\ F(P_k,j)\ +\sum_{n=P_1}^{P_N}\sum_{i(n)}{}^{*}C_{k_j m_i}(P_0,P_1)\ F(n,i)$$

$$\sum_{m=P_1}^{P_M}\sum_{i(m)}{}^{*}C_{k_j m_i}(P_0,P_1)F(m,i)+\sum_{n=P_1}^{P_N}\sum_{i(n)}B^T_{ok_j}\ G'_{kn}\ B_{on_i}\ F(n,i)$$

$$\sum_{m=P_1}^{P_M}\sum_{i(m)}B^T_{ok_j}\ G'_{kJ}\ B_{om_i}\ F(m,i) \qquad (6.4\text{-}6.k)$$

When we consider the set of equations 6.4-5.k' and 6.4-6.k we find that we can compact these equations in the form

$$D = (C_0+\bar{C}_1+\bar{G}+C_1+C_2+\ldots+C_M)\ F$$

(6.4-7)

in which D and F are column matrices having first the pertinent points of the N polypeds and then the pertinent

points of the M polypeds as matrix elements. The C_0 matrix is the same as in previous cases, a diagonal matrix composed of the flexibility matrices of each of the external members of each of the polypeds. The C_1 through C_M matrices are the appropriate matrices for a chain of polypeds, bordered at the top and left by N rows and columns of zero matrix elements. The $\bar{G}$ and $\bar{C}_1$ matrices are a bit more complicated; they are the appropriate matrices for a loop of polypeds bordered on the right and below by M columns and rows of matrix segments. Let us display them in partitioned form.

$$\bar{G} = B_o^T \begin{bmatrix} G'_{kn} & G'_{kJ} \\ & \\ G'_{Jn} & G'_{JJ} \end{bmatrix} B_o \qquad (6.4\text{-}8)$$

in which, as before, o denotes an arbitrary point and

$$B_o = \operatorname{diag}\left[B_{o1}, \ldots, B_{ok}, \ldots, B_{oN}, B_{o1'}, \ldots, B_{oM}\right]$$

(6.4-8.1)

and each of the B_{oi} is in turn a diagonal matrix having an element for each external member end in that i-th polyped. And for $\bar{C}_1$ we have

$$\bar{C}_1 = \begin{bmatrix} {}^*C_{kn}(P_0,P_1) & {}^*C_{kJ}(P_0,P_1) \\ & \\ {}^*C_{Jn}(P_0,P_1) & {}^*C_{JJ}(P_0,P_1) \end{bmatrix} \qquad (6.4\text{-}9)$$

in the above equations both n and k take on all values

from 1 to N. The partitioned matrices G'_{kn} and $^*C_{kn}(P_0,P_1)$ are N x N matrices; the partitioned matrices G'_{JJ} and $^*C_{JJ}(P_0,P_1)$ are M x M matrices. The J is the particular nucleus to which the chain of polypeds is attached.

Both the $\overline{G}$ and $\overline{C}_1$ matrices are augmented by adding rows and columns of matrix elements to the G and C_1 matrices of the simple loop of polypeds. We divide the rows (columns) into two segments, those corresponding to the polypeds in the loop of polypeds and those corresponding to the polypeds in the chain of polypeds. The portions of the added rows (columns) corresponding to the polypeds in the loop of polypeds have the same elements as the J-th row (column) of the G or C_1 matrix. The portion of the added rows (columns) which correspond to the polypeds in the chain of polypeds have as elements the element in the G or C_1 matrix which is in the J-th row and J-th column.

Thus we have the solution for this combined loop-chain in terms of the elements in the individual chain of polypeds and loop of polypeds. Knowing these simple elements we can immediately write the appropriate terms of this combined loop-chain.

We could go on and enumerate other types of combinations of simple solutions or combinations of complex ones, however, there is really no need to do this. With the several combined solutions we have already we are prepared to go forth and make the most complex structure degenerate into combinations

of loops and chains and chain-loops and loop-chains for which we can immediately write the solutions.

CHAPTER 7

Overview of the Algebraic Solution

7. This chapter is only intended to give an insight into some of the problems of solution of the set of equations and some possible ways to prepare the set of equations to aid in solution. Also we must consider the boundary conditions and other restraints necessary to solve the physical problem.

7.1. Boundary Conditions.

In all the cases considered, the "fixed" or constrained ends were all concentrated in one polyped, usually in one member. In the physical problem at hand we may find that one (or more) external member in each polyped is actually constrained. Thus there are several $D_{P_i}(j)$ which are zero or have some fixed value. Also there may be some constraints which allow only certain types of forces to be applied, ie, a pipe hanger which provides only a vertical component of force and no moment components. If certain components of a constraint force must vanish, then the corresponding components of the appropriate $F(P_i, j)$ must be zero.

We may also have a situation in which there is a constraint expressed in terms of several pertinent points, such as having several external ends attach to a common point. Then the sum of the forces on the external ends meeting at this point are zero, and the displacements are all the same. The statement of such conditions could well be considered as subsidiary equations to the set of equations.

7.2. Categorizing the Set of Equations.

In a real physical problem after we have substituted the boundary conditions, and considered any other subsidiary equations, such as the sum of several forces being zero, we will find that exactly half of the components of the two column matrices D and F are known. We can divide the set of equations into several categories, each of which has some particular properties and its own most profitable method of solution. These categories are:

1. All the components of the various forces are known, and all the components of the displacements are unknown.

2. All the components of the various displacements are known, and all the components of the forces are unknown.

3. At some pertinent points all the force components are known and all the displacement components are unknown. At other pertinent points the opposite is true.

4. At some pertinent points a "mixed" situation exists; that is, if a force component is known, then the corresponding displacement component is unknown, and vice versa. It is the nature of our real physical problem that never do we have a situation where a force component and the corresponding displacement component are both either known or unknown.

5. The situations arising out of subsidiary equations, such as having several external members joined at a common point. If we substitute these subsidiary conditions into the set of equations we will find that the resulting set of equations will belong to one of the above four categories.

In the first category "solution" is accomplished merely by matrix multiplication. In the first three categories we can solve the set of matrix equations considering the 6 dimensional F and D matrices as scalar type elements, subject to a few obvious restrictions such as lack of general commutivity. However, in the fourth category we must consider each component of the F and D matrices as separate scalar quantities.

In the third and fourth categories we can separate the known and unknown components by a suitable pre- and post-multiplication of the large flexure matrix S , using a matrix N which is a unit matrix with its rows suitably scrambled so that in the transformation $N^T S N$ we have

$$D = S F \qquad (7.2\text{-}1)$$

$$D' = N^T S N F' \qquad (7.2\text{-}2)$$

in which now we have

$$D' = \left\{ D_{k_1}, D_{k_2}, \ldots\ldots, D_{k_r}, D_{u_1}, \ldots\ldots, D_{u_s} \right\} \qquad (7.2\text{-}2.1)$$

$$F' = \left\{ F_{u_1}, F_{u_2}, \ldots\ldots, F_{u_r}, F_{k_1}, \ldots\ldots, F_{k_s} \right\} \qquad (7.2\text{-}2.2)$$

where the subscript k refers to the various known components and the subscript u refers to the various unknown components of the forces or displacments.

When the known and unknown quantities are separated

there are several methods available to solve this set of equations. We leave it to the algebraist or computer programmer to choose which is most suitable or is already available in the cumputer subroutine library.

CHAPTER 8

Alternate Point of View- Multianchor Structure

8. In some structures it may be advantageous to consider a slightly different point of view. That is, one in which we consider the whole structure with its multiple fixed points, allow for thermal gradients in members, and have the results expressed in terms of forces and displacements at the various junction points within the structure.

8.1. Definitions and Usages.

Since our point of view is different it will be advantageous to consider some slight modifications in previous definitions and usages.

8.1.1. The same polyped as defined in section 2.3 will be used in this chapter; however, the external ends of members will be designated as O_k if that end is "fixed" and will be designated as Q_j if the end is a junction point (joins to another polyped) or is a free end (not joined to any other member or fixed point). The point which is the nucleus of the polyped P_i will be designated, as before, P_i.

8.1.2. We will number the P_i and Q_j and O_k points serially throughout the structure. It will be convenient to have the following usages.

P_i •——• Q_j means that there is a member which links from the point P_i to the point Q_j. (Recall that point Q_j is not a fixed point).

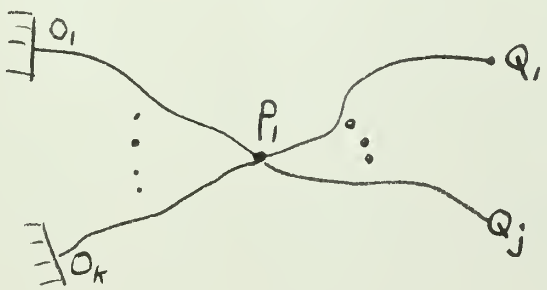
(8.1-1)

$P_i \bullet\!\!\!-\!\!\!/\!\!/\!\!\!-\!\!\!\bullet Q_j$ means that there is no member which links from the point P_i to the point Q_j.

(8.1-2)

8.2. One Polyped.

Let us first consider one polyped of the structure. Consider the following sketch for the ensuing development.



We have for the i-th member of the polyped P_1

$$D(P_1) = C(i,P_1)\ F(i,P_1) + B^T_{iP_1}\ D(i) + K(i,P_1)$$

(8.2-1)

in which $C(i,P_1)$ and $F(i,P_1)$* have the same meaning as in equations 2.6-C and 2.6-D. The matrices D and F are column matrices defined in section 2.1.

$C(i,P_1)\ F(i,P_1)$ represents the displacement of the member extending from the i-th end to the point P_1. The force is considered to act on the member at the point P_1, while the end i is considered fixed.

$B^T_{iP_1}\ D(i)$ represents the translation of the motion of the point i, considering the member as rigid.

$K(i,P_1)$ represents the motion of the end P_1 with respect to the end i due to any thermal strains in the member.

There are as many equations 8.2-1 as there are members of the polyped P_1, that is, members attaching to the point P_1.

$$i = O_1, O_2, O_3, \ldots, O_k, Q_1, Q_2, \ldots, Q_j$$

Rearranging the terms in equation 8.2-1 we have

$$F(i,P_1) = C(i,P_1)^{-1}\left[D(P_1) - \left(B^T_{iP_1}\ D(i) + K(i,P_1)\right)\right] \tag{8.2-2}$$

As a convenience in the algebraic manipulation to follow

let $$D'(i) = B^T_{iP_1}\ D(i) + K(i,P_1) \tag{8.2-3D}$$

(the designation of the polyped may be added as a subscript if needed for clarity). Then equation 8.2-2 becomes

$$F(i,P_1) = C(i,P_1)^{-1}\left(D(P_1) - D'(i)\right) \tag{8.2-4}$$

In order for the point P_1 to be in equilibrium, the sum of all the forces acting on P_1 must be zero. If there is an external force acting <u>on</u> the point P_1 we designate it $F_{P_1}(P_1)$. Then

$$\sum_{P_1} F(i,P_1) + F_{P_1}(P_1) = 0 \tag{8.2-5}$$

where the $\sum\limits_{P_1}$ indicates the summation over <u>all</u> members of

the polyped P_1. Now substituting equation 8.2-4 into equation 8.2-5

$$\sum_{P_1} C(i,P_1)^{-1} D(P_1) - \sum_{P_1} C(i,P_1)^{-1} D'(i) + F_{P_1}(P_1) = 0$$

(8.2-6)

Rearranging equation 8.2-6 and solving for $D(P_1)$

$$D(P_1) = \left[\sum_{P_1} C(i,P_1)^{-1}\right]^{-1} \cdot \left(\sum_{P_1} C(i,P_1)^{-1} D'(i) - F_{P_1}(P_1)\right)$$

(8.2-7)

Now in particular if in equation 8.2-1, $i = Q_j$

$$D(P_1) = C(Q_j,P_1)\, F(Q_j,P_1) + D'_{P_1}(Q_j)$$

(8.2-8)

Equating equations 8.2-7 and 8.2-8

$$C(Q_j,P_1)\, F(Q_j,P_1) = -\, D'_{P_1}(Q_j)$$

$$+ \left[\sum_{P_1} C(i,P_1)^{-1}\right]^{-1} \cdot \left(\sum_{P_1} C(i,P_1)^{-1} D'(i) - F_{P_1}(P_1)\right)$$

(8.2-9)

Now let us split the summation over all members of the polyped P_1 into the summation over the fixed members of the polyped and the summation over the other members of the polyped.

Let

$\sum\limits_{P_1}^{o}$ denote the summation over the fixed members of the polyped P_1

and $\sum_{P_1-Q_j}$ denote the summation over the other members of the polyped P_1. That is those members of the polyped P_1 such that $P_1 \bullet\!\!-\!\!\bullet Q_j$. (Recall that the points Q_j are not fixed points).

Then equation 8.2-9 becomes

$$C(Q_j,P_1)\ F(Q_j,P_1) = \left[\sum_{P_1} C(i,P_1)^{-1}\right]^{-1} \cdot \left(\sum_{P_1}^{\circ} C(i,P_1)\ D'(i)\right)$$

$$+ \left[\sum_{P_1} C(i,P_1)^{-1}\right]^{-1} \left[\sum_{P_1-Q_j} C(i,P_1) D'(i) - F_{P_1}(P_1)\right] - D'_{P_1}(Q_j)$$

(8.2-10)

Let us define the following symbol, $T(P_1)$. Note that it is a constant of the polyped P_1 and is independent of the forces and displacements of the structure.

$$T(P_1) = \left[\sum_{P_1} C(i,P_1)^{-1}\right]^{-1} \cdot \sum_{P_1}^{\circ} C(i,P_1)\ D'(i)$$

(8.2-11D)

or if we substitute equation 8.2-3D

$$T(P_1) = \left[\sum_{P_1} C(i,P_1)^{-1}\right]^{-1} \cdot \sum_{P_1}^{\circ} C(i,P_1) \left(B^T_{iP_1} D(i) + K(i,P_1)\right)$$

(8.2-11.1)

The various D(i) in the above equation are prescribed. They are the displacement, if any, of "fixed" ends.

Let us also define the symbol $b_{P_1}(Q_j)$, for all possible values of Q_j.

$$b_{P_1}(Q_j) = \left[\sum_{P_1} C(i,P_1)^{-1}\right]^{-1} C(Q_j,P_1) \qquad (8.2\text{-}12D)$$

in which $C(Q\ ,P\ ) \equiv 0$ if $P_1 \bullet\!\!\not\!\!-\!\!\bullet Q_j$, that is,

the flexure matrix is nonzero only for actual members of the structure. It is zero for those members of the structure which could theoretically exist but do not actually exist. That is, $b_{P_1}(Q_j)$ is nonzero only for those Q_j which belong to the polyped P_1. The definition of a symbol over all possible values and then allowing only certain classes of the symbol to have nonzero value is used throughout this chapter.

Using the definitions in equations 8.2-11D and 8.2-12D, equation 8.2-10 becomes

$$C(Q_j,P_1)\ F(Q_j,P_1) = T(P_1) - D'_{P_1}(Q_j) + \sum_{k=Q_1}^{Q_J} b_{P_1}(k)\ D'(k) - \left[\sum_{P_1} C(i,P_1)^{-1}\right]^{-1} F_{P_1}(P_1) \qquad (8.2\text{-}10.1)$$

Let us consider the translation of a force. Writing equation 2.1.6-3 in the present notation we have

$$F(P_1,Q_j) = - B_{Q_jP_1} F(Q_j,P_1) \qquad (8.2\text{-}13)$$

and both forces act on the member $\overline{P_1Q_j}$. Now if we substitute equation 8.2-10.1 into 8.2-13

$$F(P_1,Q_j) = - B_{Q_jP_1} C(Q_j,P_1)^{-1}\left[T(P_1) - D'_{P_1}(Q_j) + \sum_{k=Q_1}^{Q_K} b_{P_1}(k)\ D'(k) - \left[\sum_{P_1} C(i,P_1)^{-1}\right]^{-1} F_{P_1}(P_1)\right] \qquad (8.2\text{-}14)$$

There are matrix equations of this form for all the Q_j of polyped P_1 such that $P_1 \frown Q_j$. That is there are as many equations as there are members of the polyped with either a free end or link to another polyped.

Since it has served its purpose in simplifying the algebra so far, let us substitute for the $D'(i)$ from equation 8.2-3D. Then equation 8.2-14 becomes

$$F(P_1,Q_j) = - B_{Q_jP_1} C(Q_j,P_1)^{-1} \Big[T(P_1) - B^T_{Q_jP_1} D(Q_j) - K(Q_j,P_1) + \sum_{k=Q_1}^{Q_J} b_{P_1}(k)\, B^T_{kP_1} D(k) + \sum_{k=Q_1}^{Q_J} b_{P_1}(k)\, K(k,P_1) - \Big[\sum_{P_1} C(i,P_1)^{-1}\Big]^{-1} F_{P_1}(P_1) \Big]$$

(8.2-14.1)

Let us group together all the terms which contain only parameters of the members of the polyped, ie, $C(X,P_1)$, $K(X,P_1)$, etc, into the one term. Let us also group together the parameters associated with the $D(k)$.

$$A_{P_1}(Q_j,0) = B_{Q_jP_1} C(Q_j,P_1)^{-1} \Big[T(P_1) - K(Q_j,P_1) + \Big[\sum_{P_1} C(i,P_1)^{-1}\Big]^{-1} C(k,P_1)\, K(k,P_1) - \Big[\sum_{P_1} C(i,P_1)^{-1}\Big]^{-1} F_{P_1}(P_1)$$

(8.2-15.0)

$$A_{P_1}(Q_j,k) = B_{Q_jP_1} C(Q_j,P_1)^{-1} \Big[b_{P_1}(k)\, B^T_{kP_1} \Big] \qquad k \neq Q_j$$

or substituting for $b_{P_1}(k)$

$$A_{P_1}(Q_j,k) = B_{Q_jP_1} C(Q_j,P_1)^{-1}\left[\sum_{P_1} C(i,P_1)^{-1}\right]^{-1} C(k,P_1)\, B^T_{kP_1}$$

$$\text{for } k \neq Q_j \qquad (8.2\text{-}15.k)$$

$$A_{P_1}(Q_j,Q_j) = B_{Q_jP_1} C(Q_j,P_1)^{-1}\left[b_{P_1}(Q_j) - I\right] B^T_{Q_jP_1}$$

$$= B_{Q_jP_1} C(Q_j,P_1)^{-1}\left(\left[\sum_{P_1} C(i,P_1)^{-1}\right]^{-1} C(Q_j,P_1) - I\right) B^T_{Q_jP_1}$$

$$(8.2\text{-}15.Q_j)$$

Then we have

$$-F(P_1,Q_j) = A_{P_1}(Q_j,0) + \sum_{k=Q_1}^{Q_J} A_{P_1}(Q_j,k)\, D(k) \qquad (8.2\text{-}16)$$

This is true for those Q_j such that $P_1 \bullet\!\!-\!\!\bullet Q_j$. It will be convenient to extend this equation for use when $P_1 \bullet\!\!\#\!\!\bullet Q_j$. In such cases obviously $F(P_1,Q_j) = 0$ since there is no member $\overline{P_1Q_j}$, and we will use the convention that

$$A_{P_1}(Q_j,x) \equiv 0 \qquad \text{for } x = 0, Q_1, Q_2, \ldots, Q_J$$

$$\text{when } P_1 \bullet\!\!\#\!\!\bullet Q_j \qquad (8.2\text{-}17D)$$

Using this convention equation 8.2-16 has meaning for all Q_j.

Equation 8.2-16 is a relationship between the force on a member of the polyped P_1 at the junction point Q_j and the displacements of the other junction points. There is nothing in the derivation of this equation particular to any

one polyped and thus we can immediately generalize this equation for use in a polyped P_i.

$$- F(P_i,Q_j) = A_{P_i}(Q_j,0) + \sum_{k=Q_1}^{Q_J} A_{P_i}(Q_j,k)\ D(k) \tag{8.2-18}$$

8.3. Interconnected Polypeds.

Now that we have the force on one member at one junction point Q_j in terms of the displacements at the other junction points let us consider several polypeds, say M polypeds, connected at J different junction points. We must have an equilibrium of forces at each of the junction points. Let us consider some junction point, say Q_j, For the equilibrium of forces we have

$$\sum_{m=P_1}^{P_M} F(m,Q_j) + F(0,Q_j) = 0 \tag{8.3-1}$$

where $F(0,Q_j)$ is a force exerted <u>on</u> the junction point directly, not through the action of any member. Normally this force is zero, but it is included here for completeness. Substituting in equation 8.3-1 the appropriate equation 8.2-18 for each $m = P_1, P_2, \ldots, P_M$.

$$- F(0,Q_j) + \sum_{m=P_1}^{P_M} \left[A_m(Q_j,0) + \sum_{k=Q_1}^{Q_J} A_m(Q_j,k)\ D(k) \right] = 0 \tag{8.3-2}$$

Recall from equation 8.2-17D that for the various values of m

$A_m(Q_j,k) \equiv 0$, unless for $m = P_m$ we have $P_m \bullet\!\!-\!\!\!-\!\!\bullet Q_j$ $(k = Q_1, \ldots, Q_J)$.

In a finite summation we can interchange the order of summation, thus equation 8.3-2 may be written

$$- F(0,Q_j) + \sum_{m=P_1}^{P_M} A_m(Q_j,0) + \sum_{k=Q_1}^{Q_J} \left[\sum_{m=P_1}^{P_M} A_m(Q_j,k) \right] D(k) = 0 \tag{8.3-3}$$

There are $j = 1, 2, \ldots, J$ such equations.

Performing the summation over m and denoting the sum as follows:

$$S_{j0} = \sum_{m=P_1}^{P_M} A_m\,(Q_j,0) \tag{8.3-4D}$$

$$S_{jk} = \sum_{m=P_1}^{P_M} A_m\,(Q_j,k) \tag{8.3-5D}$$

Then the equation 8.3-3 becomes

$$- F(0,Q_j) + S_{j0} + \sum_{k=Q_1}^{Q_J} S_{jk}\, D(k) = 0 \tag{8.3-6}$$

$$j = 1, 2, \ldots, J$$

Note that $S_{jk} \equiv 0$ unless there is a polyped P_m such that

$P_m \longleftrightarrow Q_j$ and $P_m \longleftrightarrow Q_k$. For if $P_m \nleftrightarrow Q_k$ then

$b_{P_m}(Q_k) = 0$ and consequently in equation 8.2-15.k

$$A_{P_m}(Q_j,Q_k) = 0 \,. \tag{8.3-7.1}$$

Also if $P_m \nleftrightarrow Q_j$ in equation 8.2-17D then all the

$$A_{P_m}(Q_j,x) = 0. \tag{8.3-7.2}$$

We are thus in a position to solve the set of equations 8.3-6 for the various $D(k)$, the displacements of the junct-

ion points. The $F(0,Q_j)$ are presumably known forces on the junction points and all the A's can be computed from the equations 8.2-15 as functions of the polyped members, i.e., flexure matrices, thermal gradient coefficients, and the geometry of the structure.

Thus we have described the multianchor structure in a manner such that we can determine the displacement of all joints in the structure. It is important to note that the number of equations 8.3-6 depends only on the number of junction points and does not depend on the number of "fixed" points in the structure. The solution actually depends on both the number of junction points and the number of "free" endpoints. Considering that we have the equations in the form of equation 7.2-2, then the major portion of the solution is to solve the first r equations for the various F_u 's, then we may solve the remaining s equations by simple matrix multiplication for the various D_u 's. Usually the number of "free" endpoints is relatively few and also these points are not important engineeringwise.

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APPENDIX I

Matrix Algebra

I1. Introduction.

In this thesis only elementary matrix algebra is used. The purpose of this appendix is to define the notation used and to develop several matrix algebraic expressions of a general nature which will be referred to in the main body of the thesis.

I2. Notation.

I2.1. Matrices will be denoted by capital letters, their elements by the same lower case letter with appropriate subscripts all enclosed in parenthesis. For convenience partitioned matrices will often be used; that is, matrices whose elements are themselves matrices. These elements will be denoted by capital letters. Subject to a few obvious restrictions such as lack of general commutivity, partitioned matrices may be treated as if their elements were ordinary scalar quantities.

I2.2. Matrix addition and subtraction will be denoted by plus and minus signs; matrix multiplication will be denoted by writing the matrices in juxtaposition.

I2.3. The matrix operation of transposition will be denoted by the superscript ^{T}.

$$[a_{ij}]^T = [a_{ji}] \qquad (I2.3\text{-}1)$$

I2.4. The symbols I and 0 will denote respectively the unit or identity matrix and the null matrix. The order of

these matrices will be apparent from the context.

I2.5. The matrix operation of inverse will be denoted by a supercript $^{-1}$.

$$A\ A^{-1} = I = A^{-1}\ A \qquad (I2.5\text{-}1)$$

I2.6. When a matrix is either exhibited in extensio or when only the general or typical term is given, the elements will be enclosed in square brackets. Partitioned matrices whose elements are themselves matrices will be enclosed also in square brackets. In order to conserve space we will use braces to indicate a column matrix. Also we will use the letters "diag" followed by the principal diagonal terms enclosed in square brackets to indicate a diagonal matrix. (In a diagonal matrix only the principal diagonal terms may be nonzero; all other terms are zero).

I3. Matrix Algebra.

The following is a summary of some basic matrix algebra operations. In general, these are given without proof and the reader is referred to a standard work on the subject[6] for details of proofs.

I3.1. If A is an $(m \times n)$ matrix and B is an $(r \times s)$ matrix, the product AB has meaning only if $n = r$, and then these matrices are said to be conformable. The result $C = AB$ is an $(m \times s)$ matrix, where

$$c_{ij} = \sum_{k} a_{ik}\ b_{kj} \qquad (I3.1\text{-}1)$$

I3.2. In general matrices are not commutative, even if

conformable.

$$AB \neq BA \tag{I3.2-1}$$

I3.3. For any matrix A

$$AI = IA = A \tag{I3.3-1}$$

and $AO = OA = O$ (I3.3-2)

I3.4. When a matrix product is transposed the order of the factors forming the product is reversed. This process holds for any number of factors.

Thus, if $P = A\ B\ C$ then $P^T = C^T\ B^T\ A^T$

(I3.4-1)

I3.5. Reversal of order of products when matrices are inverted. This process has meaning only when the factors in the matrix product are all square, nonsingular matrices. This reversal rule applies to any number of factors in the product. Thus

if $P = A\ B\ C$ then $P^{-1} = C^{-1}\ B^{-1}\ A^{-1}$

(I3.5-1)

Writing the result in a particular form we have

$$A\ B\ C = (C^{-1}\ B^{-1}\ A^{-1})^{-1} \tag{I3.5-2}$$

I3.6. The inverse of the inverse of a matrix, if it exists, is the matrix itself.

$$(C^{-1})^{-1} = C \tag{I3.6-1}$$

I3.7. The transpose of the transpose of a matrix is the matrix itself.

$$(C^T)^T = C \tag{I3.7-1}$$

I3.8. If the matrix is symmetrical, then $U^T A U$ is also symmetrical; where U is any matrix conformable to A.

I3.9. The inverse of a matrix by partitioning.

Let A denote a square matrix of order m, partitioned as shown below with $r + s = m$.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{I3.9-1}$$

in which a_{11} is an r x r matrix and a_{22} is an s x s matrix. Let the inverse, $B = A^{-1}$, also be partitioned

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \tag{I3.9-2}$$

in which b_{11} is an r x r matrix and b_{22} is an s x s matrix. Since $A B = I$, we have

$$I = a_{11}b_{11} + a_{12}b_{21} \tag{I3.9-3.1}$$

$$0 = a_{21}b_{11} + a_{22}b_{21} \tag{I3.9-3.2}$$

$$0 = a_{11}b_{12} + a_{12}b_{22} \tag{I3.9-3.3}$$

$$I = a_{21}b_{12} + a_{22}b_{22} \tag{I3.9-3.4}$$

If we let

$$X = a_{22}^{-1} a_{21} \tag{I3.9-4.1}$$

$$Y = a_{12} a_{22}^{-1} \tag{I3.9-4.2}$$

$$Q = (a_{11} - a_{12}a_{22}^{-1}a_{21}) \qquad (I3.9\text{-}4.3)$$

The equations I3.9-3 can be solved using the equations I3.9-4 and we have for B

$$B = A^{-1} = \begin{bmatrix} Q^{-1} & -Q^{-1}Y \\ -XQ^{-1} & (a_{22}^{-1} + XQ^{-1}Y) \end{bmatrix} \qquad (I3.9\text{-}5)$$

4. Derivation of some useful matrix equations.

There are several simplifications of rather complex matrix terms which will be derived in detail. In the body of the thesis the reader is referred to these results. The following are derived using general terms, however in application specific substitutions must be made which in many cases will be complicated matrix expressions.

I4.1. Consider the simplification of the expression $A(A+B)^{-1}B$. Using equation I3.5-2 we have

$$A(A+B)^{-1}B = \left[B^{-1}(A+B)A^{-1}\right]^{-1} = \left[(B^{-1}A + I)A^{-1}\right]^{-1}$$

$$A(A+B)^{-1}B = (B^{-1} + A^{-1})^{-1} \qquad (I4.1\text{-}1)$$

also $B(A+B)^{-1}A = (B^{-1} + A^{-1})^{-1}$ (I4.1-2)

I4.2. Consider the simplification of the expression $A(I - (A+B)^{-1}A)$ or $A - A(A+B)^{-1}A$

$$A - A(A+B)^{-1}A = A\left[(A+B)^{-1}(A+B) - (A+B)^{-1}A\right]$$

$$A - A(A+B)^{-1}A = A(A+B)^{-1}(A+B-A)$$

(I4.2-1)

Using the results of equation I4.1-1 we have

$$A - A(A+B)^{-1}A = (B^{-1} + A^{-1})^{-1} \qquad (I4.2\text{-}2)$$

I4.3. Consider the simplification of the expression $A(A+B+C)^{-1}C$. Using equation I3.5-2 we have

$$A(A+B+C)^{-1}C = \left[C^{-1}(A+B+C)A^{-1}\right]^{-1}$$

performing the indicated multiplication we have

$$A(A+B+C)^{-1}C = (C^{-1} + C^{-1}BA^{-1} + A^{-1})^{-1} \qquad (I4.3\text{-}1)$$

I4.4. Consider the simplification of the expression $A - (A+B)(A+B+C)^{-1}A$ or $\left[I - (A+B)(A+B+C)^{-1}\right]A$.

$$A - (A+B)(A+B+C)^{-1}A = \left[(A+B+C)(A+B+C)^{-1} - (A+B)(A+B+C)^{-1}\right]A$$

$$= \left[\Big(A+B+C - (A+B)\Big)(A+B+C)^{-1}\right]A$$

$$A - (A+B)(A+B+C)^{-1}A = C(A+B+C)^{-1}A \qquad (I4.4\text{-}1)$$

Using the results of equation I4.3-1 we have

$$A - (A+B)(A+B+C)^{-1}A = (C^{-1} + A^{-1} + A^{-1}BC^{-1})^{-1} \qquad (I4.4\text{-}2)$$

I4.5. Consider the simplification of the expression $A - A(A+B+C)^{-1}(A+B)$ or $A\left[I - (A+B+C)^{-1}(A+B)\right]$.

$$A - A(A+B+C)^{-1}(A+B) = A\left[(A+B+C)^{-1}(A+B+C) +\right.$$

$$- (A+B+C)^{-1}(A+B)\Big]$$

$$= A(A+B+C)^{-1}\Big((A+B+C) - (A+B)\Big)$$

$$A - A(A+B+C)^{-1}(A+B) = A(A+B+C)^{-1}C \qquad \text{(I4.5-1)}$$

Using the results of equation I4.3-1 we have

$$A - A(A+B+C)^{-1}(A+B) = (C^{-1}+A^{-1}+C^{-1}BA^{-1})^{-1} \qquad \text{(I4.5-2)}$$

I5. Derivation of some useful relationships concerning the translation of flexure matrices.

I5.1. Using the notation of section 2.6, following the same line of reasoning as in section 2.2.1, in the more sophisticated notation we have

$$C(P_1,P_2) = B^T_{P_1P_2}\, C(P_2,P_1)\, B_{P_1P_2} \qquad \text{(I5.1-1)}$$

I5.2. Let us consider the following notation for simplicity

$$C(X,P)_n = B^T_{Pn}\, C(X,P)\, B_{Pn} \qquad \text{(I5.2-1D)}$$

Note that if $n = P$, we have

$$C(X,P)_P = C(X,P) \qquad \text{(I5.2-2)}$$

This is the flexure matrix for member $\overline{XP}$ translated to the point P. The order of the flexure indices within the parentheses is important.

If we let $n = X$ in equation I5.2-1D, then we have

$$C(X,P)_X = B^T_{PX}\ C(X,P)\ B_{PX}$$

using equation I5.1-1 we have

$$C(X,P)_X = C(P,X) \qquad (I5.2\text{-}3)$$

Let us consider the translational transitivity property of $C(X,P)_n$ translated to the point m. Using the definition I5.2-1D we have

$$\left(C(X,P)_n\right)_m = B^T_{nm} \left(B^T_{Pn}\ C(X,P)\ B_{Pn}\right) B_{nm} \qquad (I5.2\text{-}4)$$

Let $$\left(C(X,P)_n\right)_m = C(X,P)_m \qquad (I5.2\text{-}5D)$$

since the point n is in effect a dummy variable when the equation is written out in detail and equations 2.1.5-8 and 2.1.5-9 are applied as needed. Thus

$$C(X,P)_m = B^T_{nm}\ C(X,P)_n\ B_{nm} \qquad (I5.2\text{-}6)$$

I5.3. Let us consider the transpose of translated flexure matrices, say $C(X,P)^T$. Using equation I5.2-1D

$$\left(C(X,P)_n\right)^T = \left(B^T_{Pn}\ C(X,P)\ B_{Pn}\right)^T \qquad (I5.3\text{-}1)$$

Using equation I3.4-1 we have

$$= B^T_{Pn}\ C(X,P)^T \left(B^T_{Pn}\right)^T \qquad (I5.3\text{-}2)$$

Simplifying using equations 2.1.4-2 and I3.7-1

$$\left(C(X,P)_n\right)^T = B^T_{Pn}\ C(X,P)\ B_{Pn} = C(X,P)_n \qquad (I5.3\text{-}3)$$

Thus a translated flexure matrix is symmetric, as of course,

it should be.

I5.4. The following material is applicable to the development in chapter 4. The simplifications are applicable to a situation in which we have a loop of linked polypeds, that is, the point P_1 is linked to the "fixed" point and P_2 and P_n, the point P_2 is linked to the point P_1 and P_3, and so on. Using the notation

$$(\Sigma C)_{P_n} = C(P_1,P_n)_{P_n} + C(P_1,P_2)_{P_n} + \dots + C(P_{n-1},P_n)_{P_n}$$

(I5.4-1D)

from equation I5.2-2 we have

$$C(P_1,P_n)_{P_n} = C(P_1,P_n)$$

$$C(P_{n-1},P_n)_{P_n} = C(P_{n-1},P_n)$$

Thus equation I5.4-1D becomes

$$(\Sigma C)_{P_n} = C(P_1,P_n) + C(P_1,P_2)_{P_n} + \dots + C(P_{n-1},P_n)_{P_n}$$

(I5.4-1.1)

Also we have

$$\left((\Sigma C)_{P_n}\right)^{-1} = \Big[C(P_1,P_n) + C(P_1,P_2)_{P_n} + C(P_2,P_3)_{P_n} + \dots$$

$$+ C(P_{n-1},P_n)\Big]^{-1} \qquad \text{(I5.4-2D)}$$

I5.4.1. Consider the simplification of $B_{(n-1)n} (\Sigma C)_{P_n}^{-1} B^T_{(n-1)n}$

Using equation I2.5-1 for inverting the product of matrices we have

$$B_{(n-1)n} (\Sigma C)_{P_n} B^T_{(n-1)n} = \left[B^{T\,-1}_{(n-1)n} \left((\Sigma C)_{P_n}\right)^{-1} B^{-1}_{(n-1)n} \right]^{-1}$$

(I5.4-3)

Using equations 2.1.5-6 and 2.1.5-7 we have

$$= \left[B^T_{n(n-1)} (\Sigma C)_{P_n} B_{n(n-1)} \right]^{-1}$$

(I5.4-4)

If we expand the terms of $(\Sigma C)_{P_n}$ and apply equation I5.2-6 to each term we have the result

$$= \left[C(P_1,P_n)_{P_{(n-1)}} + C(P_1,P_2)_{P_{(n-1)}} + \right.$$

$$\left. \dots + C(P_{n-1},P_n)_{P_{n-1}} \right]^{-1}$$

((I5.4-5)

Applying equation I5.2-3 we have

$$= \left[C(P_1,P_n)_{P_{n-1}} + C(P_n,P_{n-1}) + \right.$$

$$\left. C(P_1,P_2)_{P_{n-1}} + \dots + C(P_{n-2},P_{n-1})_{P_{n-1}} \right]^{-1}$$

(I5.4-6)

Note that all the flexure indices are written with the first index as the end of the member nearest to the fixed point of the structure and the second index as the end of the member nearest to the point to which the flexure matrices are translated.

Thus if we use equation I5.4-1D we have

$$\left((\Sigma C)_{P_{n-1}}\right)^{-1} = B_{(n-1)n}\left((\Sigma C)_{P_{n-1}}\right)^{-1} B^T_{(n-1)n} \tag{I5.4-7}$$